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“Тухайн уламжлалт дифференциал тэгшитгэлийн алгебр ба аналитик судалгаа”

Суурь судалгааны төслийн тайлан

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“Тухайн уламжлалт дифференциал тэгшитгэлийн алгебр ба аналитик судалгаа” (2019-2022)

Гарчиг

1. Төслийн ерөнхий мэдээлэл

- Төслийн нэр: Тухайн уламжлалт дифференциал тэгшитгэлийн алгебр ба аналитик судалгаа,
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- Төслийн багийн бүрэлдхүүн

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2. Судалгааны зорилго, түүний биелэлт

“Тухайн уламжлалт дифференциал тэгшитгэлийн алгебр ба аналитик судалгаа” сэдэвт суурь судалгааны төсөл нь математик физикийн үндсэн тэгшитгэлүүдийн шийдийн чанарын талаарх судалгааны төсөл бөгөөд

1. Квант механикийн үндсэн тэгшитгэлүүдийн нэг болох Шредингерийн тэгшитгэлийн шийдийн талаар
2. Хугацаагаар авсан уламжлал нь бутархай зэрэгтэй байх дулааны болон долгионы тэгшитгэлүүдийн шийдийн талаар

гэсэн 2 үндсэн чиглэлтэй болно. Төслийн хугацаанд чиглэл тус бүрээр бие даасан судалгааны семинаруудыг тогтмол явуулж, ахисан түвшний оюутнуудыг уг судалгаанд хамруулж байсан. Мөн судалгааны явцын үр дүнгүүдийг тухай бүрт нь эрдэм шинжилгээний хурлуудад илтгэж байсан бөгөөд тэдгээрийг нэгтгэн олон улсын өндөр зэрэглэлийн сэтгүүлүүдэд хэвлүүлсэн юм. Төслийн явцад математик физикийн загваруудыг судлах, түүнийг бодит амьдралаас үүдэн гарсан бодлогуудыг шийдэхэд хэрэглэх, оюутнуудын хичээлээр олж авсан мэдлэгээ батжуулах, хэрэглэж сурах үйл явцыг дэмжих үүднээс “Хэрэглээний математик” эрдэм шинжилгээний хурлууд, үйлдвэрлэлтэй хамтарсан судалгааны семинарыг зохион байгуулахад төслийн багийн гишүүд гар бие оролцсоноос гадна санхүүгийн дэмжлэг үзүүлж байлаа.

Эхний чиглэлийн судалгааны чиглэл болох Шредингерийн тэгшитгэлийн шийд, харгалзах операторын спектр ба сарнилын бодлогын талаарх судалгааг төсөл эхлэхэд төсөөлж байснаас илүү өргөтгөсөн бөгөөд тухайлбал хугацаанаас хамаарсан потенциал бүхий Шредингерийн тэгшитгэлийн шийдийн хувьд Адиабатик теорем биелэхгүй байх жишээг гаргасан юм.

Төслийн тайланг үйл ажиллагаа болон үр дүнг цаг хугацааны дараалаар нэгтгэж гарган, эцэст нь гол үр дүнгүүдийг дурьдсан болно. Мөн санхүүгийн тайланг нэмж оруулсан болно. Хавсралтад хэвлүүлсэн өгүүллүүд эх хувь, хурлын хөтөлбөрүүд болон санхүүгийн баримтуудыг хавсаргалаа.

Бүлэг 3

Судалгааны үр дүн

3.1 Шредингерийн операторын спектр, сарнилын бодлогууд, шийдийн чанар

3.1.1 Регуляр потенциал бүхий Шредингерийн операторoор цэгэн харилцан үйлчлэлтэйг дөхөх

Шредингерийн операторын сарнилын бодлогын гол асуудал түүний долгионы операторын оршин байх, гүйцэт байх эсэх тухай байдаг. Олон биеийн бодлогын хувьд энэ асуудлыг 1990-ээд онд Волкер Энс нарын эрдэмтэд шийдсэн юм. Үүнээс цаашлаад долгионы операторын чанаруудын талаарх (тухайлбал L^p-зааглагдах эсэх) судалгааг хэмжээс болон потенциалын төрөл, чанараас хамааруулан олон судлаачид хийж байна [17], [2], [4]. Бид дараах ажлаараа цэгэн харилцан үйлчлэл бүхий Шредингерийн операторын долгионы операторыг энгийн Шредингерийн операторын долгионы операторын хязгаар хэлбэрээр илэрхийлж болохыг $\mathbb{R}^3$ дээрх N биеийн бодлогын хувьд баталж харуулсан болно. Үүнтэй холбоотой ажлыг [1] ажилд авч үзсэн байдаг.

$\mathbb{R}^3$ дээр $Y = \{y_1, \ldots, y_N\}$ гэсэн ялгаатай N цэг авья. $\mathcal{H} = L^2(\mathbb{R}^3)$ огторгуйд цэгэн харилцан үйлчлэл бүхий Шредингерийн оператор $H_{\alpha,Y}$ нь y_j цэг бүр дээр $\alpha_j \in \mathbb{R}$, $j = 1, \ldots, N$ параметраар өгөгдөх захын нөхцөлөөр тодорхойлогдсон бөгөөд резольвент адилтгалын тусламжтайгаар бичвэл

$$(H_{\alpha,Y} - z^2)^{-1} = (H_0 - z^2)^{-1} + \sum_{j,\ell=1}^{N} (\Gamma_{\alpha,Y}(z)^{-1})_{j\ell}\, \mathcal{G}_z^{y_j} \otimes \overline{\mathcal{G}_z^{y_\ell}}, \tag{3.1}$$

байдаг гэе. Энд $H_0 = -\Delta$ нь чөлөөт Шредингерийн оператор, $z \in \mathbb{C}^+ = \{z \in \mathbb{C} | \Im z > 0\}$, $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{R}^N$. Мөн энд $\Gamma_{\alpha,Y}(z)$ нь

$$\Gamma_{\alpha,Y}(z) \ := \ \Big(\Big(\alpha_j - \frac{iz}{4\pi}\Big)\delta_{j\ell} - \mathcal{G}_z(y_j - y_\ell)\hat{\delta}_{j\ell}\Big)_{j,\ell=1,\ldots,N}, \tag{3.2}$$

хэлбэртэй, элементүүд $z \in \mathbb{C}$-ээс хамаарсан голоморф функцүүд байх $N \times N$ хэмжээтэй симметр матриц юм. Энд $\delta_{j\ell}$ нь Кронекерийн символ, мөн

$$\mathcal{G}_z(x) = \frac{e^{iz|x|}}{4\pi|x|} \text{ ба } \mathcal{G}_z^y(x) = \frac{e^{iz|x-y|}}{4\pi|x-y|} \tag{3.3}$$

$(H_0 - z^2)^{-1}$ операторын цөм юм. Тэгвэл долгионы оператор $W^{\pm}_{\alpha,Y}$ defined by the limits

$$W^{\pm}_{\alpha,Y} u = \lim_{t \to \pm\infty} e^{itH_{\alpha,Y}} e^{-itH_0} u, \quad u \in \mathcal{H} \tag{3.4}$$

нь орших ба гүйцэт буюу $\mathrm{Image}\, W^{\pm}_{\alpha,Y} = \mathcal{H}_{ac}$ байна. Энд $\mathcal{H}_{ac}$ нь $\mathcal{H}$-д харгалзах $H_{\alpha,Y}$ дэд огторгуй.

Регуляр потенциал бүхий Шредингерийн операторыг

$$\overline{H}_Y(\varepsilon) = -\Delta + \sum_{i=1}^{N} \frac{\lambda_i(\varepsilon)}{\varepsilon^2} V_i \left(\frac{x - y_i}{\varepsilon}\right), \tag{3.5}$$

гэж тодорхойльё.

Тэгвэл $\varepsilon \to 0$ үед дээрх операторуудад харгалзах долгионы операторууд хэрхэн холбогдохыг дараах үр дүнгээр баталж үзүүлсэн болно.

Теорем 1. *Бид потенциал функцүүд дараах нөхцлүүдийг хангадаг гэе:*

(1) $V_1, \ldots, V_N$ *нь бодит функцүүд бөгөөд* $p < 3/2$, $q > 3$ *хувьд*

$$\langle x \rangle^2 V_j \in (L^p \cap L^q)(\mathbb{R}^3), \quad j = 1, \ldots, N \tag{3.6}$$

байг.

(2) *$\lambda_1(\varepsilon), \dots, \lambda_N(\varepsilon)$ нь $\varepsilon > 0$-ээс хамаарсан, бодит C^2-ангийн функц ба*

$$\lambda_j(0) = 1, \quad \lambda_j'(0) \neq 0, \quad \forall j = 1, \dots, N.$$

байг.

(3) *$H_j = -\Delta + V_j$, $j = 1, \dots, N$ операторын 0 энерги нь резонанстай байг.*

Тэгвэл дараах өгүүлбэрүүд үнэн байна:

(a) *$\overline{H}_Y(\varepsilon)$ нь $\varepsilon \to 0$ үед Y дээр байрлах, $\alpha = (\alpha_1, \dots, \alpha_N)$ параметрээс хамаардаг цэгэн харилцан үйлчлэл бүхий $H_{\alpha,Y}$ Шредингерийн оператор руу резольвентын хүчтэй дөхөлтийн утгаар буюу*

$$\lim_{\varepsilon \to 0} (\overline{H}_Y(\varepsilon) - z^2)^{-1} u = (H_{\alpha,Y} - z^2)^{-1} u, \quad \forall u \in \mathcal{H}, \tag{3.7}$$

байна.

(b) *Wave operators for the pair $(\overline{H}_Y(\varepsilon), H_0)$ хосын хувьд тодорхойлогдсон $W^{\pm}_{Y,\varepsilon}$ долгионы оператор*

$$W^{\pm}_{Y,\varepsilon} u = \lim_{t \to \pm\infty} e^{it\overline{H}_Y(\varepsilon)} e^{-itH_0} u, \quad u \in \mathcal{H} \tag{3.8}$$

нь хүчтэй дөхөлтийн утгаар орших ба гүйцэт байна. $W^{\pm}_{Y,\varepsilon}$ оператор нь

$$\lim_{\varepsilon \to 0} \| W^{\pm}_{Y,\varepsilon} u - W^{\pm}_{\alpha,Y} u \|_{\mathcal{H}} = 0, \quad u \in \mathcal{H}. \tag{3.9}$$

тэнцэтгэлийг хангана.

3.1.2 Адиабатик теорем биелэхгүй байх нэгэн тохиолдлын талаар

Хугацаанаас хамаарсан потенциал бүхий Шредингерийн тэгшитгэлийн шийд нь урт хугацаанд ямар чанартай байх нь сонирхолтой бодлогуудын нэг билээ. Хугацаанаас хамаардаг потенциал нь үетэй функц байх эсвэл хангалттай хугацааны дараа нөлөө нь багасдаг зэрэг олон тохиолдлуудыг авч үздэг юм . Үүн дотроос анхны нөхцлийн функцийг Гауссын тархалттай төстэй, экспоненциал буурдаг хувийн функцээр сонгосон тохиолдолд Шредингерийн тэгшитгэлийн шийд нь хангалттай урт хугацаанд анхны нөхцлийн функцийн чанараа хадгалдаг гэсэн үр дүнг Адиабатик дөхөлтийн теоремоор Т.Като [12] баталсан байдаг. Уг теоремыг эхлээд томьёольё.

$\mathcal{H}$ нь Гильберт огторгуй ба $\{H(t)\colon -a < t < a\}$ нь $\mathcal{H}$-д тодорхойлогдсон өөртөө хосмог операторын бүл байг. $0 < \varepsilon \ll 1$ бага параметрээс хамаарсан

$$i\varepsilon\partial_t u(t) = H(t)u(t) \tag{3.10}$$

Шредингерийн тэгшитгэлийн шийд нь цор ганц унитар групп $U_\varepsilon(t,s)$-г тодорхойлдог гэе. $\lambda(t)$ ба $\varphi(t)$ нь харгалзан $H(t)$ операторын тусгаарлагдсан хувийн утга, нормчлогдсан хувийн функц буюу

$$H(t)\varphi(t) = \lambda(t)\varphi(t), \quad \|\varphi(t)\|^2 = 1, \quad -a < t < a$$

байг. Борн-Фок ба Kato [3], [12] нарын үр дүн болох Адиабатик дөхөлтийн теорем ёсоор Шредингерийн тэгшитгэлийн

$$i\varepsilon\partial_t u_\varepsilon(t) = H(t)u_\varepsilon(t), \quad u_\varepsilon(0) = \varphi(0), \tag{3.11}$$

Кошийн бодлогын шийд $u_\varepsilon(t) = U_\varepsilon(t,0)\varphi(0)$ нь $0 < \varepsilon \to 0$ үед хувийн функцийн ε орчинд $-\delta < t < \delta$, $0 < \delta < a$ хугацааны туршид орших буюу өөрөөр хэлбэл

$$\|u_\varepsilon(t) - e^{-i\varepsilon^{-1}\int_0^t \lambda(s)ds}\varphi(t)\| \le C_\delta \varepsilon \tag{3.12}$$

байна. Энд $\|\cdot\|$ нь $L^2(\mathbb{R})$ огторгуйн норм.

Одоо $\lambda(t)$ хувийн утга нь $t \le -L$ үед $H(t)$-ийн тусгаарлагдсан хувийн утгаас тасралтгүй спектрийн рүү шилжээд, хэсэг хугацаанд буюу $-L \le t \le L$ үед тасралтгүй спектр дотор(embedded eigenvalue) байж байгаад, дахин $t > L$ үед тусгаарлагдсан хувийн утга рүү шилжих тохиолдолд адиабатик теорем хүчинтэй болохыг S.Teufel [15] баталсан байдаг. Тэгвэл өмнөх тохиолдолд $-L \le t \le L$ хугацааны туршид $H(t)$-ийн хувийн утга алга болж, зөвхөн тасралтгүй спектр үлддэг бол шийд ямар байх талаар ямар нэг үр дүн байхгүй юм. Бид энэ тохиолдлыг дараах бодогддог жишээгээр авч үзэхэд Адиабатик теорем биелдэггүй болох нь харагдсан юм.

$$i\varepsilon\partial_t u_\varepsilon = -\frac{1}{2}\partial_x^2 u_\varepsilon + V(t,x)u_\varepsilon, \quad u_\varepsilon(-L-1,x) = \varphi_0(x), \tag{3.13}$$

бодлогын хувьд потенциал нь $t < -L$, $t > L$ үед гармоник осциллятор ба $-L \leq t \leq L$ үед $V(t,x) = 0$ буюу

$$V(t,x) = \begin{cases} (t+L)^2x^2/2, & t < -L, \\ 0, & -L \leq t \leq L, \\ (t-L)^2x^2/2, & L > t, \end{cases} \tag{3.14}$$

гэе. Анхны нөхцлийн функц $\varphi_0(x) = (\pi^{\frac{1}{4}})^{-1}e^{-x^2/2}$ нь $H(-L-1) = -(1/2)\Delta + (1/2)x^2$ операторын нормчлогдсон хувийн функц юм. Энэ x^2t^2 потенциалтай Шредингерийн тэгшитгэлийн шийдийн талаар [16] ажилдаа авч үзсэн байдаг. Формал үр дүнг теорем хэлбэрээр бичье.

Теорем 2. (1)

$$m_\varepsilon(1/\varepsilon) = \frac{\pi^{-1/4}}{(\sqrt{2}e^{i/\varepsilon} + i)^{1/2}} + O(\varepsilon)\,. \tag{3.15}$$

$$l_\varepsilon^*(t) = \frac{-i\big(\kappa_\varepsilon J_{-\frac{3}{4}}(\frac{1}{8\varepsilon}) + J_{\frac{3}{4}}(\frac{1}{8\varepsilon})\big)}{-J_{-\frac{1}{4}}(\frac{1}{8\varepsilon}) + \kappa_\varepsilon J_{\frac{1}{4}}(\frac{1}{8\varepsilon})}, \tag{3.16}$$

байг. (3.13) бодлогын шийд $u_\varepsilon(\varepsilon s, x) = v_\varepsilon(s,x)$ *бол* $\varepsilon \to 0$ *үед*

$$\|v_\varepsilon(1/\varepsilon, x) - m_\varepsilon(1/\varepsilon)e^{-l_\varepsilon^*(1/\varepsilon)x^2/2}\| \leq C\varepsilon. \tag{3.17}$$

Энд $O(\varepsilon)$ *эрэмбийн гишүүн нь*

$$|m_\varepsilon(1/\varepsilon)|^4 = \pi^{-1}\big(3 + 2\sqrt{2}\sin(1/\varepsilon)\big) = \pi^{-1}\Re l_\varepsilon^*(1/\varepsilon) \tag{3.18}$$

байна.
(2) $\varphi_0(x) = \pi^{-\frac{1}{4}}e^{-x^2/2}$ *анхны нөхцлийн функцийн* $1/\varepsilon$ *хугацаанд үлдэх магадлал нь* $1/\sqrt{2} + O(\varepsilon)$ *байна.*

3.1.3 Квант орны Паули-Фиерцийн загвар

Шредингерийн тэгшитгэл нь атомын цөм ба электроны харилцан үйлчлэлийг тайлбарладаг бөгөөд үүн дээр харьцангуй сул харилцан үйлчлэл болох орны үйлчлэлийг тооцсон загварыг бодох нь математик физикийн бас нэгэн сонирхолтой бодлого хэдий ч нэлээд комплекс, ярвигтай бодлого гэж тооцогддог[10], [11], [7], [6].

Бид энэ ажилдаа чөлөөт бөөмд (электрон) цөөн тооны фотоны үйлчлэхэд энергийн шилжилт хэрхэн явагдаж байгааг харуулсан маш энгийн загварыг авч үзэж байгаа юм. Фотоны тоон дээр зааглал тавьсанаар бодлогын аналитик шийдийг олж, улмаар харгалзах операторын спектрын талаар маш тодорхой үр дүн гаргаж авсан юм. Энэ загвартай төстэй асуудлыг [13], [8], [9] ажлуудад хөндөж байсан бөгөөд бидний энэ ажлын анхны сэдлийг T.Miyao [14] өгсөн болно. Эхлээд загвараа товч танилцуулж, дараа гаргаж авсан үр дүнгээ теорем хэлбэрээр бичье. Бид

$$\mathcal{H} := L^2(\mathbb{R}^3_x) \otimes \mathcal{F}.$$

гэсэн Гильберт огторгуй авч үзэх бөгөөд $\mathcal{F}$ нь

$$\mathcal{F} = \oplus_{n=0}^{\infty} \mathcal{F}^{(n)} =: \oplus_{n=0}^{\infty} (\otimes_s^n \mathcal{S})$$

байх Фок огторгуй ба $\mathcal{S} = L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$ байг. $\otimes_s^n \mathcal{S}$-ээр $\mathcal{S}$ огторгуйн n-дахин авсан симметр тензор үржвэрийг тэмдэглэв.

$v = (v(k,1), v(k,2)) \in L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$ векторын хувьд $a(v)$, $a^*(v)$ гэсэн "annihilation"ба "creation"операторуудыг

$$a^\sharp(v) = \sum_{\lambda=1}^{2} \int a^\sharp(k,\lambda) v(k,\lambda) dk$$

байдлаар тодорхойлно. Энд a эсвэл a^*-г $a^\sharp$ гэж тэмдэглэсэн ба $a^\sharp(k,\lambda)$ нь түүний оператор цөм нь байг.

Чөлөөт фотоны энерги

$$H_f = \sum_{\lambda=1}^{2} \int \omega(k) a^*(k,\lambda) a(k,\lambda) dk,$$

гэж өгөгдөх ба $\omega(k) = |k|$ болно.

Квантчилагдсан орныг $A_g(x) = (A_{g1}(x), A_{g2}(x), A_{g3}(x))$, $x \in \mathbb{R}^3$ гэж тэмдэглэвэл

$$A_{gj}(x) = \frac{1}{\sqrt{2}} \sum_{\lambda=1}^{2} \int e_j(k,\lambda) \left[g(k) e^{-ik\cdot x} a^*(k,\lambda) + \overline{g(k)} e^{ik\cdot x} a(k,\lambda) \right] dk$$

тодорхойлогддог байг. Энд $e(k,\lambda) = (e_1(k,\lambda), e_2(k,\lambda), e_3(k,\lambda)$ нь

$$k \cdot e(k,\lambda) = 0, \quad e(k,\lambda) \cdot e(k,\mu) = \delta_{\lambda\mu}, \qquad \lambda, \mu = 1, 2$$

нөхцлүүдийг хангадаг поляризацийн вектор гэж нэрлэгдэх векторууд болог. Тэгвэл бид Паули-Фиерцийн операторыг

$$H = \frac{1}{2}(-i\nabla_x \otimes 1 - eA_g(x))^2 + 1 \otimes H_f,$$

гэж тодорхойлно. Энд e орны харилцан үйлчлэлийн констант.

Одоо тотал моментын огторгуй руу буулгавал Паули-Фиерцийн оператор хялбар хэлбэрт орох бөгөөд $\mathcal{F}$ огторгуйд

$$\bar{H}(p) \;\; = \;\; \frac{1}{2}(p - P_f - eA_g(0))^2 + H_f$$

операторыг авч үзэхэд хангалттай. Фотоны тоог 2-оор хязгаарлан, харгалзах Паули-Фиерцийн операторыг $H(p)$ гэж тэмдэглэе. Тэгвэл дараах теорем биелнэ.

Теорем 3. *(1) Дурын $p \in \mathbb{R}^3$ хувьд*

$$z_0(|p|) = \min_{k \in \mathbb{R}^3} \left\{ \frac{1}{2}(p-k)^2 + |k| + \gamma_0 \right\} = \begin{cases} \frac{1}{2}p^2 + \gamma_0, & \text{хэрэв } |p| \leq 1, \\ |p| - 1/2 + \gamma_0, & \text{хэрэв } |p| > 1. \end{cases}$$

нь $(|p|, z)$ хавтгайн муруй байг. $\gamma_0 = \frac{\pi}{1+\sigma}e^2 R^{2+2\sigma}$ ба $R > 0$ тоо нь ультраягаан үйлчлэлийн радиус. Тэгвэл $H(p)$ операторын спектр нь

$$\operatorname{spec} H(p) = \begin{cases} z^*(p) \cup [z_0(|p|), +\infty), & 0 \leq |p| < \rho \quad \text{үед }, \\ [z_0(|p|), +\infty), & \text{бусад тохиолдолд} \end{cases}$$

байна. Энд $z^*(p)$ нь $H(p)$-ын хувийн утга болно.
(2) $E_\sigma(p) = \inf H(p)$ гэвэл эффектив масс гэж нэрлэгдэх тоог

$$\frac{1}{m_{eff}} = \lim_{\sigma \to 0} \frac{\partial^2 E_\sigma(p)}{\partial |p|^2}\Big|_{p=0}.$$

гэж тодорхойльё. Паули-Фиерцийн $H(p)$ операторын хувьд

$$\frac{1}{m_{eff}} = \frac{1 - \frac{8}{3}\pi e^2 \ln(R/2+1)}{1 + \frac{8}{3}\pi e^2 \ln(R/2+1)}$$

томьёо хүчинтэй байна.

3.2 Хугацаагаар бутархай эрэмбийн уламжлалтай, хувьсах коэффициенттэй, конвекц-диффузийн тэгшитгэл, инвариант шийдүүдийн тухай

Оливер Хэвисайдын 1880-д оны үед зохиогдсон дамжуулах шугамын цахилгаан соронзон долгионы шинж чанарыг илэрхийлэх телеграфийн тэгшитгэл нь өндөр давтамжтай цахилгаан хэлхээ зэрэг цахилгааны салбарт чухал үүрэгтэй байдаг. Олон төрлийн сонгодог тухайн уламжлалт дифференциал тэгшитгэлүүдийн хугацаагаар авсан уламжлалыг бутархай эрэмбийн уламжлал болгон сольсон хувилбарууд нь тухайн физик үзэгдлүүдийг илүү сайн тайлбарладаг болох нь сүүлийн жилүүдэд олон судалгаагаар харагдаж байна. Бид энэ ажлаар

$$\begin{cases} \frac{\partial^\alpha u}{\partial t^\alpha} = v_x, \\ \frac{\partial^\alpha v}{\partial t^\alpha} = f(x)u_x + g(x)u, \end{cases} \tag{3.19}$$

тэгшитгэлийг судална. Энд бутархай эрэмбийн уламжлал нь Риман-Луивиллын тодорхойлолтоор

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} := \begin{cases} \frac{\partial^n u}{\partial t^n}, & \text{хэрэв } \alpha \in \mathbb{N}, \\ \frac{1}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial t^n} \int_0^t \frac{u(x,s)}{(t-s)^{\alpha-n+1}} ds, & \text{хэрэв } \alpha \in (n-1, n) \text{ ба } n \in \mathbb{N}. \end{cases} \tag{3.20}$$

өгөгдсөн бөгөөд $f(x)$, $g(x)$ нь хангалттай удаа дифференциалчлагддаг функцүүд байна.
Уг системийн өмнө судлагдсан байдал:

- (3.19) системийн $\alpha = 1$ болон $g(x) = 0$ тохиолдлыг Блюман, Күмей нар 1987 онд [34] ажилд Ли-гийн бүлгэн ангиллыг хийж, зарим инвариант шийдүүдийг олсон.

- 2015 онд Хуан, Шен нар [37] ажилд өмнөх зүйлд дурьдагдсан ажлын хугацаагаар бутархай эрэмбийн тохиолдлын буюу (3.19) системийн $\alpha > 0$, $g(x) = 0$ тохиолдлыг судалж Ли-гийн симметрүүдийг олж харуулсан.

- 2019 онд бид [38] ажилд $\alpha > 0$, $g(x) = 0$ тохиолдлыг судалж, Ли-гийн симметр бүлгэн ангиллыг хийж, инвариант шийдүүдийг тусгай функцээр илэрхийлж олсон. Уг ажлаар судалсан систем маань хувьсах коэффициент $f(x)$-ээс хамааран хувьсагч нь ялгагдах, хувьсагч нь ялгагдахгүй болон хувьсагч нь ялгагдаж ч болдог, ялгагдахгүй ч байж болдог 3 төрөлд ангилагдаж байгааг олж тогтоосон.

Энэ ажилд бид өмнө судалсан [38] ажлаа өргөтгөн $g(x) \neq 0$ тохиолдлыг судална. Товчхондоо, бид $f(x)$ болон $g(x)$ хувьсах функцүүдээс хамааруулан (3.19) системийн Ли-гийн бүлгэн ангиллыг хийж, ангилал болгон дахь Ли-гийн симметрүүдээс тогтох Ли алгебрийн оптимал алгебрийг олж, оптимал алгебрийн хувиргалт бүрд харгалзах инвариант шийдүүдийг Миттаг-Леффлер, өргөтгөсөн Райт, Фокс Эйч тусгай функцүүдээр илэрхийлж харуулна.

3.2.1 Хугацаагаар бутархай эрэмбийн уламжлалтай конвекц-диффузийн тухайн уламжлалт дифференциал тэгшитгэлийн Ли-гийн симметр анализ

(3.19) системийн Ли-гийн симметрүүдийг олохын тулд бид эхлээд

$$\begin{cases} \tilde{X}\left(u_{t^\alpha} - v_x\right)|_{(3.19)} = 0, \\ \tilde{X}(v_{t^\alpha} - f(x)u_x - g(x)u)|_{(3.19)} = 0. \end{cases}$$

гэсэн тодорхойлогч системийн шийдийг олох ёстой. Энд өргөтгөсөн инфинитезимал үүсгүүр нь

$$\tilde{X} = \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial t} + \mu\frac{\partial}{\partial u} + \phi\frac{\partial}{\partial v} + \mu^{(\alpha)}\frac{\partial}{\partial u_{t^\alpha}} + \mu^{(1)}\frac{\partial}{\partial u_x} + \cdots + \phi^{(\alpha)}\frac{\partial}{\partial v_{t^\alpha}} + \phi^{(1)}\frac{\partial}{\partial v_x} + \cdots, \tag{3.21}$$

гэж тодорхойлогдоно. Энд τ, ξ, μ болон ϕ нь инфинитезималууд, $\mu^{(\alpha)}$, $\mu^{(n)}$, $\phi^{(\alpha)}$ болон $\phi^{(n)}$ $(n = 1, 2, \ldots)$ нь [40]-д өгөгдсөн өргөтгөсөн инфинитезималууд байна. Тодорхойлогч системийг задалж бичвэл

$$\begin{cases} \left(\mu^{(\alpha)} - \phi^{(1)}\right)\Big|_{(3.19)} = 0, \\ \left(\phi^{(\alpha)} - f_x\xi u_x - f\mu^{(1)} - g_x u\xi - g\mu\right)\Big|_{(3.19)} = 0. \end{cases} \tag{3.22}$$

болно. Дээрх системийн хоорондоо шугаман хамааралгүй $D_t^{\alpha-n}u$, $D_t^{\alpha-n}v$, $D_t^{\alpha-n}u_x$, $D_t^{\alpha-n}v_x$, v_x, u_x, v_t, u_xv_t, v_xv_t, u_xv_x, v_x^2 гэсэн тухайн уламжлалуудын болон тогтмол тоо буюу 1-ийн өмнөх коэффициентүүдийг бүгдийг нь тус тус тэгтэй тэнцүүлбэл (3.22) систем нь

$$\binom{\alpha}{n}\frac{\partial^n \mu_u}{\partial t^n} - \binom{\alpha}{n+1} D_t^{n+1}(\tau) = 0, \qquad n = 1, 2, \ldots, \tag{3.23}$$

$$\frac{\partial^n \mu_v}{\partial t^n} = 0, \qquad n = 1, 2, \ldots, \tag{3.24}$$

$$D_t^n(\xi) = 0, \qquad n = 1, 2, \ldots, \tag{3.25}$$

$$\mu_u - \alpha D_t(\tau) - \phi_v + \xi_x = 0, \tag{3.26}$$

$$f\mu_v - \phi_u = 0, \tag{3.27}$$

$$\frac{\partial^\alpha \mu}{\partial t^\alpha} - u\frac{\partial^\alpha \mu_u}{\partial t^\alpha} - v\frac{\partial^\alpha \mu_v}{\partial t^\alpha} + gu\mu_v - \phi_x = 0, \tag{3.28}$$

$$\frac{\partial^n \phi_u}{\partial t^n} = 0, \qquad n = 1, 2, \ldots, \tag{3.29}$$

$$\binom{\alpha}{n}\frac{\partial^n \phi_v}{\partial t^n} - \binom{\alpha}{n+1} D_t^{n+1}(\tau) = 0, \qquad n = 1, 2, \ldots, \tag{3.30}$$

$$f\phi_v - \alpha f D_t(\tau) - f_x\xi - f\mu_u + f\xi_x = 0, \tag{3.31}$$

$$\frac{\partial^\alpha \phi}{\partial t^\alpha} - v\frac{\partial^\alpha \phi_v}{\partial t^\alpha} - u\frac{\partial^\alpha \phi_u}{\partial t^\alpha} + gu\phi_v - \alpha gu D_t(\tau)$$
$$-g_xu\xi - g\mu - f\mu_x = 0, \tag{3.32}$$

$$\tau_x = \tau_u = \tau_v = 0, \tag{3.33}$$

$$\xi_u = \xi_v = 0. \tag{3.34}$$

болно. Энд, бид μ, ϕ инфинитезималуудыг u болон v хувьсагчдаар шугаман гэж үзнэ. (3.23)-(3.34) тэгшитгэлүүдэд анализ хийснээр бид (3.19) системийг дараах 2 дүгнэлтийг хийж болно.

1. (3.19) систем нь дурын хувьд дараах симметрүүдтэй байна

$$X_0 = c_1(x,t)\frac{\partial}{\partial u} + c_2(x,t)\frac{\partial}{\partial v} \quad \text{болон} \quad X_1 = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}.$$

   Энд, $c_1(x,t)$, $c_2(x,t)$ нь (3.19) системийн дурын шийд байна. Мэдээж X_0 симметр нь анхны системийн шугаман чанарыг илэрхийлж байна.

2. f, g коэффициент функцүүдийн дараах тохиолдлуудад (3.19) систем нь нэмэлт Ли-гийн симметртэй байна. (3.23)-(3.34) тэгшитгэлүүдэд анализ хийснээр f, g коэффициент функцүүд нь $f_x - 2g$ илэрхийлэл 0-тэй тэнцэх эсэхээс хамааран (3.19) дараах хоёр тохиолдолд хуваагдана.

Тохиолдол 1

$f_x - 2g \neq 0$ гэж үзье. Энэ тохиолдолд, (3.23)-(3.34) тэгшитгэлүүдэд анализ хийсний дүнд дараах хоёр дэд тохиолдлууд гарна.

Тохиолдол 1.1

$\left(\sqrt{f(x)}\right)' - \frac{g(x)}{\sqrt{f(x)}} \neq 0$ тохиолдол авч үзье. f болон g функцүүд нь

$$g(x) = \frac{\lambda_2 f^{\frac{1}{2}}}{\lambda_1 + \int f^{-\frac{1}{2}} dx} + \frac{f_x}{2},\ \lambda_1, \lambda_2 \in \mathbb{R},\ \lambda_2 \neq 0$$

хангадаг байна. Тэгвэл дурын $f(x)$ болон $g(x) = \frac{\lambda_2 f^{\frac{1}{2}}}{\lambda_1 + \int f^{-\frac{1}{2}} dx} + \frac{f_x}{2}$ (энд $\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_2 \neq 0$) функцийн хувьд (3.19) систем нь X_0, X_1-ээс гадна

$$X_2 = f^{\frac{1}{2}} \left(\lambda_1 + \int f^{-\frac{1}{2}} dx\right) \frac{\partial}{\partial x} + \frac{t}{\alpha} \frac{\partial}{\partial t} - \frac{f_x}{2 f^{\frac{1}{2}}} \left(\lambda_1 + \int f^{-\frac{1}{2}} dx\right) u \frac{\partial}{\partial u}.$$

нэмэлт симметрүүдтэй байна.

Тохиолдол 1.2

$\left(\sqrt{f(x)}\right)' - \frac{g(x)}{\sqrt{f(x)}} \neq 0$ тохиолдол авч үзье. Тэгвэл дурын $f(x)$ функц болон

$$g(x) = \lambda_2 f^{\frac{1}{2}} + \frac{f_x}{2},\ \lambda_2 \in \mathbb{R},\ \lambda_2 \neq 0$$

функцийн хувьд (3.19) систем дараах нэмэлт симметрүүдтэй байна

$$X_3 = f^{\frac{1}{2}} \frac{\partial}{\partial x} - \frac{f_x}{2 f^{\frac{1}{2}}} u \frac{\partial}{\partial u}.$$

Тохиолдол 2

$f_x - 2g = 0$ тохиолдлыг авч үзье. Энэ тохиолдолд (3.19) систем дараах 3 нэмэлт симметрүүдтэй байна

$$\begin{aligned} X_4 &= f^{\frac{1}{2}} \int f^{-\frac{1}{2}} dx \frac{\partial}{\partial x} + \frac{t}{\alpha}\frac{\partial}{\partial t} - \frac{f_x}{2f^{\frac{1}{2}}} \int f^{-\frac{1}{2}} dx u \frac{\partial}{\partial u}, \\ X_5 &= f^{\frac{1}{2}} \frac{\partial}{\partial x} - \frac{f_x}{2f^{\frac{1}{2}}} u \frac{\partial}{\partial u}, \\ X_6 &= f^{-\frac{1}{2}} v \frac{\partial}{\partial u} + f^{\frac{1}{2}} u \frac{\partial}{\partial v}. \end{aligned}$$

Бид энэ хэсгийн гол үр дүнгээ дараах байдлаар томъёолж болно.

Теорем 4. *(3.19) тэгшитгэлийн системээр өгөгдсөн хувьсах коэффициенттэй, бутархай эрэмбийн дифференциал тэгшитгэлийн систем авч үзье. Тэгвэл,*

i) дараах гурван тохиолдлоос ялгаатай дурын $f(x)$ болон $g(x)$ функцүүдийн хувьд (3.19) систем нь

$$X_0 = c_1(x,t)\frac{\partial}{\partial u} + c_2(x,t)\frac{\partial}{\partial v}, \quad and \ X_1 = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v},$$

гэсэн хоёр симметртэй байна. Энд, $c_1(x,t)$ болон $c_1(x,t)$ нь (3.19) системийн дурын шийд байна.

ii) дурын $f(x)$ болон $g(x) = \frac{\lambda_2 f^{\frac{1}{2}}}{\lambda_1 + \int f^{-\frac{1}{2}} dx} + \frac{f_x}{2}$ (энд $\lambda_1, \lambda_2 \in \mathbb{R}, \ \lambda_2 \neq 0$) функцүүдийн хувьд (3.19) систем нь X_0, X_1-с гадна дараах симметрүүдтэй байна

$$X_2 = f^{\frac{1}{2}} \left(\lambda_1 + \int f^{-\frac{1}{2}} dx\right) \frac{\partial}{\partial x} + \frac{t}{\alpha}\frac{\partial}{\partial t} - \frac{f_x}{2f^{\frac{1}{2}}} \left(\lambda_1 + \int f^{-\frac{1}{2}} dx\right) u \frac{\partial}{\partial u}.$$

iii) дурын $f(x)$ болон $g(x) = \lambda_2 f^{\frac{1}{2}} + \frac{f_x}{2}$ (энд $\lambda_2 \in \mathbb{R}, \ \lambda_2 \neq 0$) функцүүдийн хувьд (3.19) систем нь X_0, X_1-с гадна дараах симметрүүдтэй байна

$$X_3 = f^{\frac{1}{2}} \frac{\partial}{\partial x} - \frac{f_x}{2f^{\frac{1}{2}}} u \frac{\partial}{\partial u}.$$

iv) дурын $f(x)$ болон $g(x) = \frac{f_x}{2}$ функцүүдийн хувьд (3.19) систем нь X_0, X_1-с гадна дараах

$$
\begin{aligned}
X_4 &= f^{\frac{1}{2}}\int f^{-\frac{1}{2}}dx\frac{\partial}{\partial x} + \frac{t}{\alpha}\frac{\partial}{\partial t} - \frac{f_x}{2f^{\frac{1}{2}}}\int f^{-\frac{1}{2}}dxu\frac{\partial}{\partial u},\\
X_5 &= f^{\frac{1}{2}}\frac{\partial}{\partial x} - \frac{f_x}{2f^{\frac{1}{2}}}u\frac{\partial}{\partial u},\\
X_6 &= f^{-\frac{1}{2}}v\frac{\partial}{\partial u} + f^{\frac{1}{2}}u\frac{\partial}{\partial v}.
\end{aligned}
$$

3.2.2 Ли-гийн симметрүүдээр үүсгэгдсэн Ли алгебрийн дэд алгебруудын оптимал систем

Дифференциал тэгшитгэлийн бүлгэн анализийг хийхийн тулд төгсгөлөг хэмжээст Ли алгебрийн дэд алгебруудыг олох нь чухал байдаг. [24] ажилд дурдсан автоморфизмийн бүлгээр ижил хэмжээст дэд алгебрууд нь бие биендээ буух учир бие биендээ буудаггүй ялгаатай дэд алгебр болгоноос нэг нэг симметрийг төлөөлөгч болгон сонгон авч үүсгэсэн системийг оптимал систем гэж нэрлэнэ. 6 хүртэлх хэмжээстэй Ли алгебрийн оптимал системийг [48] ажилд олсон байдаг бөгөөд бидний судлах систем маань аль ч тохиолдолд 6-аас бага хэмжээст Ли алгебртэй симметрүүдтэй учир симметрүүдийн Ли алгебруудын оптимал системд харгалзах симметрийн инвариант шийдүүдийг олоход хангалттай юм.

Бид (3.19) системийн бүлгэн ангиллыг хийсэн учир одоо инфинитезимал симметрүүдийн Ли алгебрийн нэг хэмжээст оптимал систем болон бүлгэн инвариант шийдүүдийг олъё. Үүнээс хойшхи судалгаанд бид X_0 симметрийг тооцохгүй.

Тохиолдол ii)

Энэ тохиолдолд (3.19) систем

$$
\begin{cases}
\frac{\partial^\alpha u}{\partial t^\alpha} = v_x,\\
\frac{\partial^\alpha v}{\partial t^\alpha} = f(x)u_x + \left(\frac{\lambda_2 f(x)^{\frac{1}{2}}}{\lambda_1 + \int f(x)^{-\frac{1}{2}}dx} + \frac{f(x)_x}{2}\right)u,
\end{cases}
\tag{3.35}
$$

болно. Энд $f(x)$ хангалттай удаа дифференциалчлагдах функц, $\lambda_1, \lambda_2 \in \mathbb{R},\ \lambda_2 \neq 0$ байна. харгалзах инфинитезимал симметрүүд нь

$$\begin{aligned} X_1 &= u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}, \\ X_2 &= f^{\frac{1}{2}}\left(\lambda_1 + \int f^{-\frac{1}{2}}dx\right)\frac{\partial}{\partial x} + \frac{t}{\alpha}\frac{\partial}{\partial t} - \frac{f_x}{2f^{\frac{1}{2}}}\left(\lambda_1 + \int f^{-\frac{1}{2}}dx\right)u\frac{\partial}{\partial u} \end{aligned}$$

байна. X_1, X_2 нь Абелийн Ли алгебр үүсгэх ба оптимал систем нь

$$U_1 = X_1, \quad U_2 = X_2 + aX_1, \quad \text{энд } a \in \mathbb{R}$$

болно. U_2-д харгалзах характеристик тэгшитгэл нь

$$\frac{dx}{f^{\frac{1}{2}}\left(\lambda_1 + \int f^{-\frac{1}{2}}dx\right)} = \frac{\alpha dt}{t} = \frac{du}{\left[a - \frac{f_x}{2f^{\frac{1}{2}}}\left(\lambda_1 + \int f^{-\frac{1}{2}}dx\right)\right]u} = \frac{dv}{av} \quad (3.36)$$

болох ба эндээс I интегралууд нь $z = \left(\lambda_1 + \int f^{-\frac{1}{2}}dx\right)^{-\frac{1}{\alpha}} t$ ба

$$\begin{cases} u = f^{-\frac{1}{2}}\left(\lambda_1 + \int f^{-\frac{1}{2}}dx\right)^a \varphi(z), \\ v = \left(\lambda_1 + \int f^{-\frac{1}{2}}dx\right)^a \psi(z) \end{cases} \quad (3.37)$$

болно. (3.37)-ийг (3.35)-д орлуулснаар бид хувьсагчдын тоог бууруулж

$$\begin{cases} \frac{d^\alpha \varphi}{dz^\alpha} = a\psi - \frac{1}{\alpha}z\psi_z, \\ \frac{d^\alpha \psi}{dz^\alpha} = (a+\lambda_2)\varphi - \frac{1}{\alpha}z\varphi_z. \end{cases} \quad (3.38)$$

ердийн дифференциал тэгшитгэлийн систем гаргаж авна. U_1 симметр дараагийн тооцоонд ч мөн гарах бөгөөд ямар ч инвариант шийд өгөхгүй.

Тохиолдол iii)

Энэ тохиолдолд (3.19) систем

$$\begin{cases} \frac{\partial^\alpha u}{\partial t^\alpha} = v_x, \\ \frac{\partial^\alpha v}{\partial t^\alpha} = f(x)u_x + \left(\lambda_2 f^{\frac{1}{2}} + \frac{f_x}{2}\right)u, \end{cases} \quad (3.39)$$

болно. Энд $f(x)$ хангалттай удаа дифференциалчлагддаг функц ба $\lambda_2 \in \mathbb{R},\ \lambda_2 \neq 0$ байна. харгалзах инфинитезимал симметрүүд

$$X_1 = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}, \quad X_3 = f^{\frac{1}{2}}\frac{\partial}{\partial x} - \frac{f_x}{2f^{\frac{1}{2}}}u\frac{\partial}{\partial u}$$

болох ба коммутатор нь 0 учир өмнөх тохиолдолтой адилаар Абелийн бүлэг болно. Оптимал систем нь

$$U_1 = X_1, \quad U_3 = X_3 + aX_1, \quad \text{where } a \in \mathbb{R}$$

болно. U_3 симметрийн хувьд характеристик арга хэрэглэснээр

$$\begin{cases} u = f^{-\frac{1}{2}} \exp\left(a \int f^{-\frac{1}{2}} dx\right) \varphi(t), \\ v = \exp\left(a \int f^{-\frac{1}{2}} dx\right) \psi(t), \end{cases} \tag{3.40}$$

инвариант шийдтэй болох ба хялбарчлагдсан систем нь

$$\begin{cases} \frac{d^\alpha \varphi(t)}{dt^\alpha} = a\psi(t), \\ \frac{d^\alpha \psi(t)}{dt^\alpha} = (a + \lambda_2)\varphi(t) \end{cases} \tag{3.41}$$

болно.

Тохиолдол iv)

Энэ тохиолдолд (3.19) систем нь

$$\begin{cases} \frac{\partial^\alpha u}{\partial t^\alpha} = v_x, \\ \frac{\partial^\alpha v}{\partial t^\alpha} = f(x)u_x + \frac{f_x}{2}u, \end{cases} \tag{3.42}$$

болно. Энд $f(x)$ хангалттай удаа тасралтгүй дифференциалчлагдах функц байна. Тооцоог хялбарчлахын тулд бид

$$\begin{aligned} Y_1 = \quad -X_4 \quad &= -f^{\frac{1}{2}} \int f^{-\frac{1}{2}} dx \frac{\partial}{\partial x} - \frac{t}{\alpha}\frac{\partial}{\partial t} + \frac{f_x}{2f^{\frac{1}{2}}} \int f^{-\frac{1}{2}} dx u \frac{\partial}{\partial u}, \\ Y_2 = \quad -X_5 \quad &= -f^{\frac{1}{2}}\frac{\partial}{\partial x} + \frac{f_x}{2f^{\frac{1}{2}}}u\frac{\partial}{\partial u}, \\ Y_3 = \quad X_1 \quad &= u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}, \\ Y_4 = \quad X_6 \quad &= f^{-\frac{1}{2}}v\frac{\partial}{\partial u} + f^{\frac{1}{2}}u\frac{\partial}{\partial v}. \end{aligned}$$

$[Y_i,Y_j]$	Y_1	Y_2	Y_3	Y_4
Y_1	0	Y_2	0	0
Y_2	$-Y_2$	0	0	0
Y_3	0	0	0	0
Y_4	0	0	0	0

Хүснэгт 3.1: Y_1, Y_2, Y_3, Y_4-ийн коммутатор хүснэгт.

суурь сонгож авъя. Дээрх суурийн хувьд коммутаторыг Хүснэгт 3.1-д харуулав (энд, i, j-ээр харгалзан мөр ба баганыг дугаарлав).

Хүснэгт 3.1-ээс Ли алгебр нь [48]-ийн $A_2 \oplus 2A_1$ алгебртэй адил болох нь харагдана. Тиймээс Y_1, Y_2, Y_3, Y_4-р үүсгэгдсэн Ли алгебрийн оптимал систем нь дараах байдлаар өгөгдөнө.

$$\begin{aligned}
U_1 &= Y_3 = X_1, \\
U_4 &= Y_1 - a_1Y_3 - a_2Y_4 = -a_1X_1 - X_4 - a_2X_6, \text{ энд } a_1, a_2 \in \mathbb{R}, \\
U_5 &= Y_2 - a_1Y_3 - a_2Y_4 = -a_1X_1 - X_5 - a_2X_6, \\
&\quad \text{энд } (a_1, a_2) \in \{(\pm 1, a), (0, \pm 1), (0, 0) | a \in \mathbb{R}\}, \\
U_6 &= a_1Y_3 + Y_4 = a_1X_1 + X_6, \text{ энд } a_1 \in \mathbb{R}.
\end{aligned}$$

U_4-ийн хувьд инвариант гадаргуугийн нөхцөл

$$\begin{cases} f^{\frac{1}{2}} \left(\int f^{-\frac{1}{2}}dx\right) u_x + \frac{t}{\alpha}u_t = \left(a_1 - \frac{f_x}{2f^{\frac{1}{2}}} \int f^{-\frac{1}{2}}dx\right) u + a_2 f^{-\frac{1}{2}} v \\ f^{\frac{1}{2}} \left(\int f^{-\frac{1}{2}}dx\right) v_x + \frac{t}{\alpha}v_t = a_1 v + a_2 f^{\frac{1}{2}} u, \end{cases} \tag{3.43}$$

болох бөгөөд эндээс $z = t\left(\int f^{-\frac{1}{2}}dx\right)^{-\frac{1}{\alpha}}$. Анзац буюу хувиргалт нь

$$\begin{cases} u = f^{-\frac{1}{2}} \left(\int f^{-\frac{1}{2}}dx\right)^{a_1+a_2} \varphi(z) + f^{-\frac{1}{2}} \left(\int f^{-\frac{1}{2}}dx\right)^{a_1-a_2} \psi(z) \\ v = \left(\int f^{-\frac{1}{2}}dx\right)^{a_1+a_2} \varphi(z) - \left(\int f^{-\frac{1}{2}}dx\right)^{a_1-a_2} \psi(z) \end{cases} \tag{3.44}$$

байна. (3.44)-ийг (3.42)-д орлуулснаар

$$\begin{cases} \frac{d^\alpha \varphi}{dz^\alpha} = (a_2 + a_1)\varphi - \frac{1}{\alpha} z\varphi_z \\ \frac{d^\alpha \psi}{dz^\alpha} = (a_2 - a_1)\psi + \frac{1}{\alpha} z\psi_z \end{cases} \tag{3.45}$$

хялбарчлагдсан систем гарна. Төсөөтэйгээр, U_5-д харгалзах хувиргалт

$$\begin{cases} u = f^{-\frac{1}{2}} \exp\left[(a_1+a_2)\int f^{-\frac{1}{2}}dx\right]\varphi(t) + f^{-\frac{1}{2}} \exp\left[(a_1-a_2)\int f^{-\frac{1}{2}}dx\right]\psi(t) \\ v = \exp\left[(a_1+a_2)\int f^{-\frac{1}{2}}dx\right]\varphi(t) - \exp\left[(a_1-a_2)\int f^{-\frac{1}{2}}dx\right]\psi(t). \end{cases} \tag{3.46}$$

болох ба хялбарчлагдсан систем нь

$$\begin{cases} \frac{d^\alpha \varphi(t)}{dt^\alpha} = (a_2+a_1)\varphi(t), \\ \frac{d^\alpha \psi(t)}{dt^\alpha} = (a_2-a_1)\psi(t). \end{cases} \tag{3.47}$$

болно. U_6-д харгалзах инвариант шийд байхгүй.

3.2.3 Бүлгэн инвариант шийдүүд

Өмнөх хэсэгт олсон хялбаршуулсан тэгшитгэлүүдийн шийдийг олох замаар (3.19) системийн инвариант шийдийг олно. (3.38)-с (3.41)-д өгөгдсөн хялбаршуулсан системүүд нь

$$\begin{cases} \dfrac{d^\alpha \varphi}{dz^\alpha} = a_1\psi + \dfrac{b_1}{\alpha} z\psi', \\ \dfrac{d^\alpha \psi}{dz^\alpha} = a_2\varphi + \dfrac{b_2}{\alpha} z\varphi', \end{cases} \tag{3.48}$$

гэсэн ерөнхий хэлбэртэй байгааг харж болно. Энд, a_1, a_2, b_1, b_2 нь тогтмол тоонууд байна. $g(x) = \frac{f(x)_x}{2}$ үед (3.45), (3.47) системүүд нь

$$\frac{d^\alpha \varphi}{dz^\alpha} = a\varphi + \frac{b}{\alpha} z\varphi', \tag{3.49}$$

ерөнхий хэлбэртэй байна. Энд, a, b нь тогтмол тоонууд байна. Өөрөөр хэлбэл, (3.19)-ийн инвариант шийдийг олох бодлого нь үндсэндээ (3.48) болон (3.49) тэгшитгэлийн шийдийг олох бодлого руу шилжлээ гэсэн үг юм. (3.48), (3.49)-ийн шийдийн тухай бид [39] ажилдаа дэлгэрэнгүй дурьдсан бөгөөд [38] ажлын Лемма 2.1, 2.2-оос бид дараах үр дүнг шууд гаргаж авч болно.

Энд Миттаг-Леффлер болон өргөтгөсөн Райт функцээр илэрхийлэгдсэн шийдүүд нь $z \in \mathbb{R}$ мужид тодорхойлогдсон байгаа бол Фокс-Эйч функцээр илэрхийлэгдсэн шийдүүд нь $z > 0$ мужид тодорхойлогдсон байгааг дурдъя.

Тохиолдол i)

Энэ тохиолдолд, (3.35)-ийн шийдүүд дараах байдлаар илэрхийлэгдэнэ:

1. $0 < \alpha < 1$ үед [39]-ийн Лемма 2.1-ийг ашиглавал (3.38)-д $a_1 = a$, $a_2 = a + \lambda_2$ ба $b_1 = b_2 = 1$ гэж авбал бид

$$\begin{aligned} u(x,t) &= cf^{-\frac{1}{2}}(x)\left(\lambda_1 + \int f^{-\frac{1}{2}}dx\right)^a \varphi\left(\left(\lambda_1 + \int f^{-\frac{1}{2}}dx\right)^{-1/\alpha} t\right), \\ v(x,t) &= c\left(\lambda_1 + \int f^{-\frac{1}{2}}dx\right)^a \psi\left(\left(\lambda_1 + \int f^{-\frac{1}{2}}dx\right)^{-1/\alpha} t\right) \end{aligned} \tag{3.50}$$

өгөгдсөн инвариант шийдтэй болно. Энд,

$$\begin{aligned} \varphi(z) &= H^{2,0}_{1,2}\left[\frac{z^{-2\alpha}}{4}\middle| \begin{array}{c} (1, 2\alpha) \\ \left(\frac{a}{2}+\frac{1}{2}, 1\right), \left(\frac{a+\lambda_2}{2}, 1\right) \end{array}\right], \\ \psi(z) &= H^{2,0}_{1,2}\left[\frac{z^{-2\alpha}}{4}\middle| \begin{array}{c} (1, 2\alpha) \\ \left(\frac{a}{2}, 1\right), \left(\frac{a+\lambda_2}{2}+\frac{1}{2}, 1\right) \end{array}\right]. \end{aligned}$$

2. $\alpha \geq 1$ үед Лемма 2.1 [39]-ийн гуравдугаар хэсгийг ашиглавал бид

$$u(x,t) = cf^{-\frac{1}{2}}(x)\left(\lambda_1 + \int f^{-\frac{1}{2}}dx\right)^a \varphi\left(\left(\lambda_1 + \int f^{-\frac{1}{2}}dx\right)^{-1/\alpha} t\right), \tag{3.51}$$

$$v(x,t) = c\left(\lambda_1 + \int f^{-\frac{1}{2}}dx\right)^a \psi\left(\left(\lambda_1 + \int f^{-\frac{1}{2}}dx\right)^{-1/\alpha} t\right) \tag{3.52}$$

гэж өгөгдсөн инвариант шийдтэй болно. Энд,

$$\begin{aligned} \varphi(z) &= \sum_{k=1}^{n} c_{k,1} z^{\alpha-k} {}_3\Psi_1\left[4z^{2\alpha}\middle| \begin{array}{c} \left(\frac{a}{2}-\frac{k}{2\alpha}+1, 1\right), \left(\frac{a+\lambda_2}{2}-\frac{k}{2\alpha}+\frac{1}{2}, 1\right), (1,1) \\ (1+\alpha-k, 2\alpha) \end{array}\right] \\ &+ 2\sum_{k=1}^{n} c_{k,2} z^{2\alpha-k} {}_3\Psi_1\left[4z^{2\alpha}\middle| \begin{array}{c} \left(\frac{a}{2}-\frac{k}{2\alpha}+\frac{3}{2}, 1\right), \left(\frac{a+\lambda_2}{2}-\frac{k}{2\alpha}+1, 1\right), (1,1) \\ (1+2\alpha-k, 2\alpha) \end{array}\right], \\ \psi(z) &= 2\sum_{k=1}^{n} c_{k,1} z^{2\alpha-k} {}_3\Psi_1\left[4z^{2\alpha}\middle| \begin{array}{c} \left(\frac{a}{2}-\frac{k}{2\alpha}+1, 1\right), \left(\frac{a+\lambda_2}{2}-\frac{k}{2\alpha}+\frac{3}{2}, 1\right), (1,1) \\ (1+2\alpha-k, 2\alpha) \end{array}\right] \\ &+ \sum_{k=1}^{n} c_{k,2} z^{\alpha-k} {}_3\Psi_1\left[4z^{2\alpha}\middle| \begin{array}{c} \left(\frac{a}{2}-\frac{k}{2\alpha}+\frac{1}{2}, 1\right), \left(\frac{a+\lambda_2}{2}-\frac{k}{2\alpha}+1, 1\right), (1,1) \\ (1+\alpha-k, 2\alpha) \end{array}\right] \end{aligned}$$

ба $n \in \mathbb{N}:\ 0 \leq n-1 < \alpha \leq n$, c, $c_{k,1}$, $c_{k,2}$ $(k = 1, \ldots, n)$ нь тогтмол тоонууд байна.

3.2.4 Тохиолдол ii)

Энэ тохиолдолд Лемма 2.2 [39]-ийг $a_1 = a$ and $a_2 = a + \lambda_2$ ашигласнаар (3.39)-ийн шийд

$$\begin{aligned} u(x,t) &= f^{-\frac{1}{2}} \exp\left(a \int f^{-\frac{1}{2}} dx\right) \varphi(t), \\ v(x,t) &= \exp\left(a \int f^{-\frac{1}{2}} dx\right) \varphi(t), \end{aligned} \tag{3.53}$$

гэж олдоно. Энд

$$\begin{aligned} \varphi(t) &= \sum_{k=1}^{n} c_{k,1} t^{\alpha-k} E_{2\alpha,1+\alpha-k}\left(a(a+\lambda_2)t^{2\alpha}\right) + a \sum_{k=1}^{n} c_{k,2} t^{2\alpha-k} E_{2\alpha,1+2\alpha-k}\left(a(a+\lambda_2)t^{2\alpha}\right), \\ \psi(t) &= (a+\lambda_2) \sum_{k=1}^{n} c_{k,1} t^{2\alpha-k} E_{2\alpha,1+2\alpha-k}\left(a(a+\lambda_2)t^{2\alpha}\right) + \sum_{k=1}^{n} c_{k,2} t^{\alpha-k} E_{2\alpha,1+\alpha-k}\left(a(a+\lambda_2)t^{2\alpha}\right); \end{aligned}$$

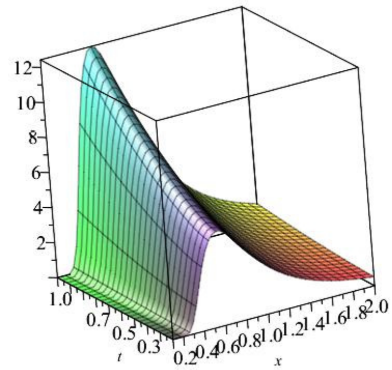
ба $n \in \mathbb{N}: \ 0 \leq n-1 < \alpha \leq n,$, $c_{k,1}$, $c_{k,2}$ $(k = 1, \ldots, n)$ нь тогтмол тоонууд байна.

$g(x) = 0$ тохиолдолд зөвхөн $f(x) = x^2$ үед л (3.19) тэгшитгэл хувьсагч нь ялгагдах тэгшитгэлд шилждэг ([38]) бол дээрх тохиолдолд дурын $f(x)$ функцийн хувьд хувьсагч нь ялгагдах тэгшитгэлд шилждэг юм. Тухайлбал, бид жишээ болгон (3.19) тэгшитгэлийн шийдийг $f(x) = x^6$, $g(x) = 3x^5 + x^3$ үед (3.53)-ээр өгөгдсөн $u(x)$ шийдийг $c_{1,2} = 0$, $c_{1,1} = 1$ үед Зураг 2.1-д дүрсэлж үзүүлэв.

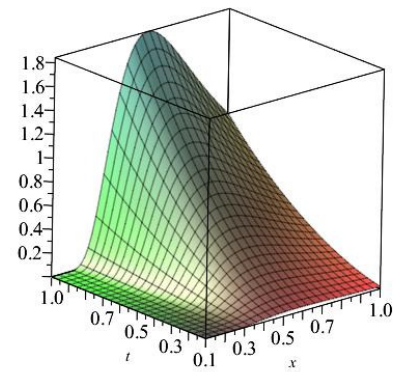
3.2.5 Тохиолдол iv)

f, g-ийн өмнөх тохиолдлуудаас ялгаатай нь дурын $f(x)$ ба $g(x) = \frac{f_x}{2}$ байх тохиолдолд инвариант шийдүүд нь бутархай эрэмбийн хоёр ердийн дифференциал тэгшитгэлүүдийн шийдүүдийн нийлбэр хэлбэрээр илэрхийлэгдэнэ. U_4-д харгалзах (3.42)-ийн инвариант шийдүүд нь дараах байдлаар иоэрхийлэгдэнэ:

1. $0 < \alpha < 1$ үед [39] ажлын Лемма 2.2 $a = a_2 - a_1$, $b = 1$ үед ашиглавал



(a) The case $\alpha = \frac{1}{4}$



(b) The case $\alpha = \frac{7}{8}$

Зураг 3.1: $f(x) = x^2,\ g(x) = 3x^5 + x^3$ үеийн (3.19) тэгшитгэлийн шийд (3.53)

инвариант шийд

$$
\begin{aligned}
u(x,t) &= cf^{-\frac{1}{2}}t^{(a_1-a_2)\alpha}\Psi\left(-\frac{\int f^{-\frac{1}{2}}dx}{t^{\alpha}};-\alpha,1+(a_1-a_2)\alpha\right),\\
v(x,t) &= -ct^{(a_1-a_2)\alpha}\Psi\left(-\frac{\int f^{-\frac{1}{2}}dx}{t^{\alpha}};-\alpha,1+(a_1-a_2)\alpha\right) \quad (3.54)
\end{aligned}
$$

болно.

2. $\alpha \geq 1$ үед (3.45)-ийн эхний тэгшитгэлд $a = a_1 + a_2$, $b = -1$ ба хоёр дахь тэгшитгэлд $a = a_2 - a_1$, $b = 1$ гэж [39] ажлын Лемма 2.2

ашиглавал U_4-д харгалзах инвариант шийдүүд нь

$$u(x,t) = f^{-\frac{1}{2}}\left(\int f^{-\frac{1}{2}}dx\right)^{a_1+a_2-1} t^{\alpha}\sum_{k=1}^{n} c_{k,1}\frac{\left(\int f^{-\frac{1}{2}}dx\right)^{\frac{k}{\alpha}}}{t^k}\varphi_k\left(\frac{t}{\left(\int f^{-\frac{1}{2}}dx\right)^{\frac{1}{\alpha}}}\right) \quad (3.55)$$

$$+f^{-\frac{1}{2}}\left(\int f^{-\frac{1}{2}}dx\right)^{a_1-a_2-1} t^{\alpha}\sum_{k=1}^{n} c_{k,2}\frac{\left(\int f^{-\frac{1}{2}}dx\right)^{\frac{k}{\alpha}}}{t^k}\psi_k\left(\frac{t}{\left(\int f^{-\frac{1}{2}}dx\right)^{\frac{1}{\alpha}}}\right),$$

$$v(x,t) = \left(\int f^{-\frac{1}{2}}dx\right)^{a_1+a_2-1} t^{\alpha}\sum_{k=1}^{n} c_{k,1}\frac{\left(\int f^{-\frac{1}{2}}dx\right)^{\frac{k}{\alpha}}}{t^k}\varphi_k\left(\frac{t}{\left(\int f^{-\frac{1}{2}}dx\right)^{\frac{1}{\alpha}}}\right) \quad (3.56)$$

$$-\left(\int f^{-\frac{1}{2}}dx\right)^{a_1-a_2-1} t^{\alpha}\sum_{k=1}^{n} c_{k,2}\frac{\left(\int f^{-\frac{1}{2}}dx\right)^{\frac{k}{\alpha}}}{t^k}\psi_k\left(\frac{t}{\left(\int f^{-\frac{1}{2}}dx\right)^{\frac{1}{\alpha}}}\right), \quad (3.57)$$

болно. Энд

$$\varphi_k(z) = {}_2\Psi_1\left[-z^{\alpha}\left|\begin{array}{c}\left(-a_1-a_2-\frac{k}{\alpha}+1,1\right),(1,1)\\(1+\alpha-k,\alpha)\end{array}\right.\right],$$

$$\psi_k(z) = {}_2\Psi_1\left[z^{\alpha}\left|\begin{array}{c}\left(-a_1+a_2-\frac{k}{\alpha}+1,1\right),(1,1)\\(1+\alpha-k,\alpha)\end{array}\right.\right].$$

байна.

[39] ажлын Лемма 2.2 ашиглавал U_5-д харгалзах инвариант шийд

$$u(x,t) = f^{-\frac{1}{2}}\exp\left((a_1+a_2)\int f^{-\frac{1}{2}}dx\right)t^{\alpha}\sum_{k=1}^{n} c_{k,1}t^{-k}\varphi_k(t)$$

$$+ \quad f^{-\frac{1}{2}}\exp\left((a_1-a_2)\int f^{-\frac{1}{2}}dx\right)t^{\alpha}\sum_{k=1}^{n} c_{k,2}t^{-k}\psi_k(t),$$

$$v(x,t) = \exp\left((a_1+a_2)\int f^{-\frac{1}{2}}dx\right)t^{\alpha}\sum_{k=1}^{n} c_{k,1}t^{-k}\varphi_k(t)$$

$$- \quad \exp\left((a_1-a_2)\int f^{-\frac{1}{2}}dx\right)t^{\alpha}\sum_{k=1}^{n} c_{k,2}t^{-k}\psi_k(t), \quad (3.58)$$

болно. Энд

$$\varphi_k(t) = E_{\alpha,1+\alpha-k}((a_1 + a_2)t^\alpha), \quad \psi_k(t) = E_{\alpha,1+\alpha-k}((a_2 - a_1)t^\alpha)$$

ба $(a_1, a_2) \in \{(\pm 1, a), (0, \pm 1), (0, 0) | a \in \mathbb{R}\}$ байна.

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Судалгааны үйл ажиллагаа, санхүү, судалгааны үр дүнгийн нэгтгэл

2019 онд гүйцэтгэсэн үйл ажиллагаа:

1. 2019 оны 1 сараас 6 сарын хугацаанд төслийн оролцогчид хамтран “Математик физикийн тэгшитгэл” нэртэй судалгааны семинарыг тогтмол явуулж, бутархай уламжлалт систем тэгшитгэлийн талаар эрдэм шинжилгээний бүтээлүүдийг уншиж, илтгэл тавилаа. Мөн Шредингерийн тэгшитгэлийн 2 ба 3 хэмжээст огторгуйд Грины функцийн чанарын талаар судалгааны ажлуудтай танилцлаа. Уг семинарт төсөлд оролцогчдоос гадна магистр, докторын оюутнууд тогтмол оролцсон болно.

2. 2019 оны 1-6 сард 3 хэмжээст огторгуйд төгсгөлөг тооны дельта функцүүдийн нийлбэрээр илэрхийлэгдэх потенциал бүхий Шредингерийн операторын долгионы операторын талаарх судалгааг хийж, энэ нь регуляр потенциал бүхий харгалзах Шредингерийн операторын долгионы операторын хязгаар хэлбэрээр илэрхийлэгдэхийг баталлаа.
3. Төсөлд оролцогчид 2019 оны 6 сарын 3-7 хооронд болсон “Үйлдвэрлэлтэй хамтарсан математик судалгааны семинар-2”-ийг зохион байгуулалцаж, IT zone компаний зүгээс тавьсан бодлогыг шийдэхэд идэвхтэй оролцлоо.
4. 2019 оны 9 сараас 12 сарын хугацаанд төслийн оролцогчид хамтран “Математик физикийн тэгшитгэл II” нэртэй судалгааны семинарыг тогтмол явуулж, бутархай уламжлалт систем тэгшитгэлийн талаар эрдэм шинжилгээний бүтээлүүдийг уншиж, илтгэл тавплаа. Мөн “Квант механикийн нэмэлт бүлгүүд” сэдэвт семинарыг тогтмол явуулж, магистрант, докторант оюутнуудыг хамрууллаа.
5. 2019 оны 9-12-р саруудад А.Галтбаяр нь Японы Гакушюин их сургуулийн профессор К.Яжиматай хамтран 3 хэмжээст огторгуйд төгсгөлөг тооны дельта функцүүдийн нийлбэрээр илэрхийлэгдэх потенциал бүхий Шредингерийн операторын долгионы операторын талаарх судалгааг дуусгаж, уг үр дүнг “Journal of Korean Mathematical Society” сэтгүүлд өгүүлэл болгон хэвлүүлэхээр явууллаа.
6. 2019 оны 11 сард төсөлд оролцогчид “Хэрэглээний математик-2019” хурлыг зохион байгуулахад оролцож, доктор З.Ууганбаяр, Д.Хонгорзул нар илтгэл тавьж, хэлэлцүүллээ.
7. Төслийн удирдагч доктор А.Галтбаяр нь Япон Улсын Гакушюин Их Сургуулийн профессор К.Яжиматай хамтран хугацаанаас хамаарсан потенциал бүхий Шредингерийн тэгшитгэлийн шийд нь хангалттай урт хугацаанд анхны нөхцлийн функцтэй ойрхон байх тухай *Адиабатик теорем*-ийг хангахгүй байдаг эсрэг жишээг гаргах оролдлого хийсэн болно. Уг ажлыг эрдэм шинжилгээний өгүүллийн хэлбэрт оруулсан.

2021 онд гүйцэтгэсэн үйл ажиллагаа:

1. 2021 оны 1 сараас 6 сарын хугацаанд төслийн оролцогчид “Ли бүлэг 1” болон “Бутархай эрэмбийн дифференциал тэгшитгэл, түүний тооцон бодох арга” нэртэй судалгааны семинаруудыг удирдан тогтмол явуулж, бутархай уламжлалт систем тэгшитгэлийн талаар эрдэм шинжилгээний бүтээлүүдийг уншиж, илтгэл тавилаа. Уг семинаруудад төсөлд оролцогчдоос гадна магистр, докторын оюутнууд тогтмол оролцсон болно.
2. 2021 оны 1 сараас 6 сарын хугацаанд сонгодог диффуз тэгшитгэлийн өргөтгөл болох advection/convection тэгшитгэлийн шийдүүдийг тусгай функцүүдийн тусламжтайгаар илэрхийлсэн. Уг үр дүнгүүд нь өмнө мэдэгдэж байсан сонгодог үр дүнг өргөтгөж байгаа юм. Шинэ үр дүнгийн шинж чанарыг судлахад тухайн шийдүүдийн графикийг дүрслэх шаардлагатай байдаг бөгөөд энэ нь мөн тухайн үр дүнг хэвлэлтэнд хүлээн авахад дөхөмтэй байдлыг бий болгодог. Энэ

ажлын хүрээнд төсөлд оролцогчид “Ердийн дифференциал тэгшитгэл-Maple систем ашиглах нь” гэсэн сурах бичгийг хэвлүүллээ. (Уг сурах бичгийн онцлог нь их, дээд сургуулийн бакалаврын түвшинд үздэг ердийн дифференциал тэгшитгэл хичээлийн уламжлалт аргуудын аль нь тохирох эсэхийг тогтоолгүйгээр Maple системийн дифференциал тэгшитгэл бодох цор ганц арга буюу dsolve командын алгоритмийн нэг үндэс болох дифференциал тэгшитгэл бодох Ли симметрийн бүлгэн хувиргалтын аргын тухай онолын товч болон уг мэдлэгийг ашиглан dsolve командыг хэрхэн үр дүнтэйгээр ашиглах тухай түлхүү оруулсан байгаа.) Судалгааны ажлын үр дүнг импакт фактор бүхий олон улсын сэтгүүлд хэвлүүлэхээр бэлдсэн юм.

3. Төсөлд оролцогчид 2021 оны 5 сарын 8-д “Математик 2021”-ийг зохион байгуулалцаж, Итали болон Польш улсын эрдэмтдийг урьж оролцуулсан бөгөөд “Conservation laws for a certain nonlinear telegraph equations” сэдэвт илтгэлийг төслийн гишүүн Д.Хонгорзул, “Generalized hypergeometric functions” сэдэвт илтгэлийг төслийн гишүүн З.Ууганбаяр нар тавилаа.
4. Төслийн гишүүн Э.Ганхөлөг нь 2021 оны 5 сарын 15-нд болсон “Хэрэглээний математик 2020” эрдэм шинжилгээний хуралд “Хугацаанаас хамаарсан потенциалтай Шредингерийн тэгшитгэлийн аналитик шийдийн талаар” илтгэлийг хэлэлцүүллээ. Мөн төслийн гишүүн, магистрант Э.Тэлмэн "Хэрэглээний математик 2020” эрдэм шинжилгээний хуралд "Active contour model with fractional order derivative” илтгэл, 2021 оны 11 сарын 25-нд болсон "Математик, тоон технологи 2021", эрдэм шинжилгээний хуралд ”Шредингерийн операторын сарнилын тухай” сэдэвт илтгэл тавьж хэлэлцүүллээ.
5. Төслийн гишүүд З.Ууганбаяр, Д.Хонгорзул нар “Хувьсах коэффициенттэй шугаман диффуз тэгшитгэлийн Ли бүлгээр инвариант байх шийдүүдийн талаар” сэдэвт илтгэлийг “Монгол улсын гавьяат багш Д.Шагдарын мэндэлсний 90 жилийн ой” эрдэм шинжилгээний хуралд 2021 оны 12 сарын 16-нд илтгэж хэлэлцүүлэв.
6. Төслийн гишүүн Д.Хонгорзул 2021 оны 11 сарын 8-19-ний өдрүүдэд Хөгжиж буй орнуудын эмэгтэй эрдэмтдийн байгууллагын VI чуулга уулзалтанд оролцож, олон улсын эрдэм шинжилгээний хуралд “Интегро-сплайн аргын тусламжтайгаар эргэлтийн энкодертой роботын байршлыг тодорхойлох нь” сэдвээр ханан илтгэл тавьж хэлэлцүүлэв.
7. Хэдийгээр ковидын нөхцөл байдлын улмаас гадаадаас зочин профессор урьж, хамтарсан судалгаа хийх ажлыг гүйцэтгэж чадаагүй боловч өндөр хөгжилтэй (Япон, Англи зэрэг) орнуудын эрдэмтэн, профессоруудтай зайнаас харилцан, удаашралтай боловч хамтарсан судалгааны ажлаа тасралтгүйгээр явуулж байна. Уг ажлын хүрээнд төслийн гишүүн З.Ууганбаяр 2021 оны 6 сарын 30-ны өдөр “Connecting Isolated Mathematical Researchers” сэдэвт Египет, Нигер, АНУ, Их Британи зэрэг орнуудын математикч эрдэмтдийн оролцсон семинарт өөрийн судалгааны ажил болоод ковидын үеийн Монгол математикч эрдэмтдийн нөхцөл байдлын талаар танилцуулж, хэлэлцүүлэв.
8. “Хэрэглээний математик 2021” эрдэм шинжилгээний хурал ковидын нөхцөл байдлаас шалтгаалан 2022 оны эхний улиралд зохиогдохоор хойшлогдсон бөгөөд тус хуралд төслийн гишүүд 2-3 илтгэл хэлэлцүүлэхээр бэлтгэж байна.
9. Төслийн гишүүд З.Ууганбаяр, Д.Хонгорзул нар “Solutions on convection-advection equations” ажлыг хэвлэлд өгөхөд бэлэн болгож, сэтгүүлд өгөх шатандаа явж байна.
10. Төслийн гишүүд Д.Даянцолмон, А.Галтбаяр нар Web of Science мэдээллийн сангийн бүртгэлтэй, 0.4 импакт фактор бүхий “Hokkaido Mathematical Journal” сэтгүүлийн 2021 оны Vol.50, No.3 дугаарт “Non-realtivistic Pauli-Fierz Hamiltonian for less than two photons ” өгүүлийг хэвлүүллээ.

2022 онд гүйцэтгэсэн үйл ажиллагаа:

1. Төслийн оролцогчид 2022 оны 1 сараас 6 сарын хугацаанд “Numerical analysis of differential equations” болон “Ли бүлгийн дифференциал тэгшитгэл дэх хэрэглээ” нэртэй судалгааны семинаруудыг удирдан тогтмол явуулж, бутархай уламжлалт систем тэгшитгэлийн тооцон бодох арга болон ердийн болон тухайн уламжлалт дифференциал тэгшитгэлийн Ли бүлгэн

хувиргалтын талаар эрдэм шинжилгээний бүтээлүүдийг уншиж, илтгэл тавилаа. Уг семинаруудад төсөлд оролцогчдоос гадна магистр, докторын түвшний оюутнууд тогтмол оролцсон болно.

2. Төслийн оролцогчид 2022 оны 1 сараас 6 сарын хугацаанд диффуз, долгионы тэгшитгэлийн өргөтгөл болох хугацаагаар бутархай эрэмбийн уламжлалтай, хувьсах коэффициенттэй диффуз-долгионы тэгшитгэлийн Ли бүлгэн анализийг хувьсах коэффициентээс нь хамааруулан хийж, боломжтой тохиолдлуудад аналитик инвариант шийдүүдийг Фокс-Эйч, өргөтгөсөн Райт функцүүдийн тусламжтайгаар илэрхийлсэн. Хувьсах коэффициенттэй бутархай эрэмбийн диффуз-долгионы тэгшитгэл нь олон төрлийн физик системүүдийн хэлбэлзэл, хэвийн биш тархалтын процессийг загварчилдаг учир уг тэгшитгэлийн инвариант шийдүүдийг олох нь ач холбогдол өндөр юм. Зөвхөн физик процессийн загварын шийдийг өгөөд ч зогсохгүй, жинхэнэ шийдийг олсноор бутархай эрэмбийн дифференциал тэгшитгэлийг ойролцоогоор бодох аргуудыг харьцуулан сонгох замаар математикийн тооцон бодох судалгаанд ч мөн ашиглаж болох юм. Уг судалгааны ажлын үр дүнг импакт фактор бүхий олон улсын сэтгүүлд хэвлүүлэхээр бэлдэж байгаа болно.
3. Төсөлд оролцогчид 2022 оны 6 сарын 6-аас 10-ны хооронд “Үйлдвэрлэлтэй хамтарсан математик” судалгааны семинарын зохион байгуулах багт ажиллав. Уг семинарт МУИС, МГТИС, МУБИС, ХААИС, СЭЗИС, ШУА-н 40 гаруй математикч эрдэмтэн, судлаачид, оюутнууд оролцож Улаанбаатар гурил ХХК, Дижитал Солюшнс ХХК, Эрдэнэс Монгол ХХК-с тавьсан бодлогуудыг амжилттай шийдвэрлэж, шийдлийг бодлого тавьсан компаниудад танилцуулав.

Төслийн үр дүнгийн тайлан /он цагийн дараалаар/

1. Эрдэм шинжилгээний өгүүлэл (1)

Galtbayar A., Yajima K., On the approximation by regular potentials of Schrodinger operators with point interactions, *J. Korean. Math.Soc.*, Vol 57., no 20 (2020), 429-450. (Thomson Reuters-ийн JCR-IF нь 0.68-тай сэтгүүл, Хавсралт 1-ийг харна уу)

2. Сурах бичиг (1)

Д.Гантулга, З.Ууганбаяр, Д.Хонгорзул, “Ердийн дифференциал тэгшитгэл-*Maple систем ашиглах нь*”, 2020 он. (Хавсралт 2-ийг харна уу),

3. Эрдэм шинжилгээний өгүүлэл (1)

Galtbayar A., Jensen A., Yajima K., “A solvable model of the breakdown of the adiabatic approximation, J.Math.Phys. Vol 61, no. 9 (2020), (Thomson Reuters-ийн JCR-IF нь 1.3-тай сэтгүүл, Хавсралт 3-ийг харна уу)

4. Ахисан түвшний оюутны төгсөлт (1)

Э.Ганхөлөг, “Хугацаанаас хамаарсан потенциалтай Шредингерийн тэгшитгэлийн аналитик шийдийн талаар” сэдэвт хэрэглээний математикийн магистрын ажил, 2020 он. (Хавсралт 4-ийг харна уу)

5. Эрдэм шинжилгээний өгүүлэл (1)

Dayantsolmon D., Galtbayar A., "Non-realtivistic Pauli-Fierz Hamiltonian for less than two photons ", Hokkaido Mathematical Journal, Vol.50, No.3,(2021) p.309-326. (Thomson Reuters-ийн JCR-IF нь 0.4-тай сэтгүүл, Хавсралт 5-ийг харна уу))

6. Олон улсын хурлын илтгэл (1)

З.Ууганбаяр, "Pure Vs. Applied Mathematics: Problems we encounter", Workshop on Connecting Isolated Mathematical Researchers, онлайн хурал, 2021.06.30, https://www.ucl.ac.uk/~ucahrha/CIMR/

7. Олон улсын хурлын илтгэл (1)

Д.Хонгорзул, "Determining position of robot with rotary encoder based on integro-spline method", "Organization for Women in Science for the Developing World" онлайн хурал, 2021.11.08-19, (Хавсралт 6-ийг харна уу)

8. Дотоодын хурлын илтгэл (1)

З.Ууганбаяр, Д.Хонгорзул, "Хувьсах коэффициенттэй шугаман диффуз тэгшитгэлийн Лигийн бүлгээр инвариант байх шийдүүдийн талаар", Д.Шагдарын мэндэлсний 90 жилийн ойд зориулсан эрдэм шинжилгээний хурал, Улаанбаатар, 2021.12.06, (Хавсралт 7-ийг харна уу)

9. Дотоодын хурлын илтгэл (3)

- ***А.Галтбаяр, Ө.Ганхөлөг,*** "Задгай орчин дахь зэсийн баяжмалын исэлдэлтийн математик загвар", "Хэрэглээний математик 2021", Улаанбаатар, 2022.01.24.
- ***Д.Хонгорзул,*** "Хугацаагаар бутархай эрэмбийн уламжлалтай, хувьсах коэффициенттэй, шугаман диффуз-конвекцийн тэгшитгэлийн зарим инвариант шийдүүд", "Хэрэглээний математик 2021", Улаанбаатар, 2022.01.24.
- ***З.Ууганбаяр***, "On Solutions of linear time fractional telegraph equations", "Хэрэглээний математик 2021", Улаанбаатар, 2022.01.24, (Хавсралт 8-ийг харна уу)

10. Дотоодын хурлын илтгэл (2)

- **З.Ууганбаяр,** "Conservation laws for diffusion equations", "Математик 2022", Улаанбаатар, 2022.04.30
- **Д.Хонгорзул,** "Invariant solutions of convection-diffusion equations", "Математик 2022", Улаанбаатар, 2022.04.30 (Хавсралт 9-ийг харна уу.)

11.Ахисан түвшний оюутны төгсөлт (1)

Э.Тэлмэн, "Өндөр эрэмбийн Шредингерийн операторын спектрын ба сарнилын онолын талаар" сэдэвт хэрэглээний математикийн магистрын ажил, 2022 он. (Хавсралт 10-ийг харна уу)

Төслийн үр дүнгийн нэгтгэл

	Тоо	Тайлбар
Импакт фактор бүхий сэтгүүлийн өгүүлэл	3	Web of Science-ийн JCR-ийн импакт фактороор тооцсон
Эрдэм шинжилгээний хурлын илтгэл	8	Үүнээс олон улсын хуралд 2 илтгэл
Төгссөн ахисан түвшний оюутны тоо	2	
Сурах бичиг, монограф	1	
Сэтгүүлийн хянан шалгах шатандаа байгаа өгүүлэл	1	"Lie group classification of time fractional convection-diffusion equations with variable coefficient" өгүүлэл

Төслийн санхүүгийн үйл ажиллагааны тайлан

2020 оны Ковид цар тахлын улмаас 2020 онд санхүүжилт огт хийгдээгүй, үйл ажиллагаа бага явагдсаны улмаас 1 жилээр сунгагдаж, 2022 онд ихэнх санхүүжилт орж ирсэн болно. Иймээс эрдэм шинжилгээний хурал, семинар зохион байгуулах зардал, гадаад дотоод томилолтын зардал зэргийг нэлээд шахуу хугацаанд зарцуулах шаардлагатай болсон нь хүндрэл учруулсан.

	Эрдэм шинжилгээний ажлын зардлын задаргаа	Төлөвлөлт /мян.төг/	Гүйцэтгэл. /мян.төг/
1	Гэрээт ажилтнуудын ажлын хөлс, НДШ	3,500.0	3,500.0
2	Мэдээлэл худалдан авах зардал	650.0	650.0
3	Эрдэм шинжилгээний хурал, семинар, үзэсгэлэн зохион байгуулах зардал	6,500.0	6,349.1
4	Гадаадын эрдэмтэн судлаачдыг Монголд байх хугацааны үйлчилгээний зардал	8,000.0	7,985.2
5	Судалгааны ажлын тайлан бичихтэй холбогдсон зардал	3,000.0	1,234.9
6	Сэлбэг хэрэгсэл, лабораторийн хэрэгсэл худалдан авах зардал	6,750.0	6,796.5
7	Гадаад, дотоодын томилолтын зардал	3,500.0	3,503.3
8	Судалгааны тоног төхөөрөмжийн хэмжилт, суурилуулалт, засвар үйлчилгээний зардал	1,750.0	1,980.0
9	Олон улсын хурлын төлбөр	1,000.0	900.0
10	Төслийн явц, үр дүнд хяналт шинжилгээ хийх зардал /1%/	350.0	350.0
11	МУИС-ийн шимтгэл 5%	0.0	1,750.0
	Дүн	35,000.0	34,999.0

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ХАВСРАЛТ

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ON THE APPROXIMATION BY REGULAR POTENTIALS OF SCHRÖDINGER OPERATORS WITH POINT INTERACTIONS

Artbazar Galtbayar and Kenji Yajima

Abstract. We prove that wave operators for Schrödinger operators with multi-center local point interactions are scaling limits of the ones for Schrödinger operators with regular potentials. We simultaneously present a proof of the corresponding well known result for the resolvent which substantially simplifies the one by Albeverio et al.

1. Introduction

Let $Y = \{y_1, \ldots, y_N\}$ be the set of N points in $\mathbb{R}^3$ and T_0 be the densely defined non-negative symmetric operator in $\mathcal{H} = L^2(\mathbb{R}^3)$ defined by

$$T_0 = -\Delta|_{C_0^\infty(\mathbb{R}^3 \setminus Y)}.$$

Any of selfadjoint extensions of T_0 is called the Schrödinger operator with point interactions at Y. Among them, we are concerned with the ones with local point interactions $H_{\alpha,Y}$ which are defined by separated boundary conditions at each point y_j parameterized by $\alpha_j \in \mathbb{R}$, $j = 1, \ldots, N$. They can be defined via the resolvent equation (cf. [2]): With $H_0 = -\Delta$ being the free Schrödinger operator and $z \in \mathbb{C}^+ = \{z \in \mathbb{C} \,|\, \Im z > 0\}$,

$$(H_{\alpha,Y} - z^2)^{-1} = (H_0 - z^2)^{-1} + \sum_{j,\ell=1}^{N} (\Gamma_{\alpha,Y}(z)^{-1})_{j\ell}\, \mathcal{G}_z^{y_j} \otimes \overline{\mathcal{G}_z^{y_\ell}}, \tag{1}$$

where $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{R}^N$, $\Gamma_{\alpha,Y}(z)$ is an $N \times N$ symmetric matrix whose entries are entire holomorphic functions of $z \in \mathbb{C}$ given by

$$\Gamma_{\alpha,Y}(z) := \left(\left(\alpha_j - \frac{iz}{4\pi}\right)\delta_{j\ell} - \mathcal{G}_z(y_j - y_\ell)\hat{\delta}_{j\ell}\right)_{j,\ell=1,\ldots,N}, \tag{2}$$

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where $\delta_{j\ell} = 1$ for $j = \ell$ and $\delta_{j\ell} = 0$ otherwise; $\hat{\delta}_{j\ell} = 1 - \delta_{j\ell}$; $\mathcal{G}_z(x)$ is the convolution kernel of $(H_0 - z^2)^{-1}$:

$$\mathcal{G}_z(x) = \frac{e^{iz|x|}}{4\pi|x|} \quad \text{and} \quad \mathcal{G}_z^y(x) = \frac{e^{iz|x-y|}}{4\pi|x-y|}. \tag{3}$$

Since $(H_{\alpha,Y} - z^2)^{-1} - (H_0 - z^2)^{-1}$ is of rank N by virtue of (1), the wave operators $W^{\pm}_{\alpha,Y}$ defined by the limits

$$W^{\pm}_{\alpha,Y} u = \lim_{t\to\pm\infty} e^{itH_{\alpha,Y}} e^{-itH_0} u, \quad u \in \mathcal{H} \tag{4}$$

exist and are complete in the sense that $\operatorname{Image} W^{\pm}_{\alpha,Y} = \mathcal{H}_{ac}$, the absolutely continuous (AC for short) subspace of $\mathcal{H}$ for $H_{\alpha,Y}$. Wave operators are of fundamental importance in scattering theory.

This paper is concerned with the approximation of the wave operators $W^{\pm}_{\alpha,Y}$ by the ones for Schrödinger operators with regular potentials and generalizes a result in [5] for the case $N = 1$, which immediately implies that $W^{\pm}_{\alpha,Y}$ are bounded in $L^p(\mathbb{R}^3)$ for $1 < p < 3$, see remarks below Theorem 1.1. We also give a proof of the corresponding well known result for the resolvent $(H_{\alpha,Y} - z)^{-1}$ which substantially simplifies the one in the seminal monograph [2].

We begin with recalling various properties of $H_{\alpha,Y}$ (see [2]):

- Equation (1) defines a unique selfadjoint operator $H_{\alpha,Y}$ in the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^3)$, which is real and local.
- The spectrum of $H_{\alpha,Y}$ consists of the AC part $[0,\infty)$ and at most N non-positive eigenvalues. Positive eigenvalues are absent. We define $\mathcal{E} = \{ik \in i\mathbb{R}^+ \colon -k^2 \in \sigma_p(H_{\alpha,Y})\}$. We simply write $\mathcal{H}_{ac}$ and P_{ac} respectively for the AC subspace $\mathcal{H}_{ac}(H_{\alpha,Y})$ of $\mathcal{H}$ for $H_{\alpha,Y}$ and for the projection $P_{ac}(H_{\alpha,Y})$ onto $\mathcal{H}_{ac}$.
- $H_{\alpha,Y}$ may be approximated by a family of Schrödinger operators with scaled regular potentials

$$\overline{H}_Y(\varepsilon) = -\Delta + \sum_{i=1}^{N} \frac{\lambda_i(\varepsilon)}{\varepsilon^2} V_i\left(\frac{x - y_i}{\varepsilon}\right), \tag{5}$$

  in the sense that for $z \in \mathbb{C}^+$

$$\lim_{\varepsilon\to 0} (\overline{H}_Y(\varepsilon) - z^2)^{-1} u = (H_{\alpha,Y} - z^2)^{-1} u, \quad \forall u \in \mathcal{H}, \tag{6}$$

  where V_j, $j = 1, \dots, N$ are such that $H_j = -\Delta + V_j(x)$ have threshold resonances at 0 and $\lambda_1(\varepsilon), \dots, \lambda_N(\varepsilon)$ are smooth real functions of ε such that $\lambda_j(0) = 1$ and $\lambda_j'(0) \neq 0$ (see Theorem 1.1 for more details).

We prove the following theorem (see Section 4 for the definition of the threshold resonance).

Theorem 1.1. *Let Y be the set of N points $Y = \{y_1, \dots, y_N\}$. Suppose that:*

(1) $V_1, \ldots, V_N$ *are real-valued functions such that for some* $p < 3/2$ *and* $q > 3$,

$$\langle x \rangle^2 V_j \in (L^p \cap L^q)(\mathbb{R}^3), \quad j = 1, \ldots, N. \tag{7}$$

(2) $\lambda_1(\varepsilon), \ldots, \lambda_N(\varepsilon)$ *are real* C^2 *functions of* $\varepsilon \geq 0$ *such that*

$$\lambda_j(0) = 1, \quad \lambda_j'(0) \neq 0, \quad \forall j = 1, \ldots, N.$$

(3) $H_j = -\Delta + V_j$, $j = 1, \ldots, N$ *admits a threshold resonance at* 0.

Then, the following statements are satisfied:

(a) $\overline{H}_Y(\varepsilon)$ *converges in the strong resolvent sense as in* (6) *as* $\varepsilon \to 0$ *to a Schrödinger operator* $H_{\alpha,Y}$ *with point interactions at* Y *with certain parameters* $\alpha = (\alpha_1, \ldots, \alpha_N)$ *to be specified below.*

(b) *Wave operators* $W^{\pm}_{Y,\varepsilon}$ *for the pair* $(\overline{H}_Y(\varepsilon), H_0)$ *defined by the strong limits*

$$W^{\pm}_{Y,\varepsilon} u = \lim_{t \to \pm\infty} e^{it\overline{H}_Y(\varepsilon)} e^{-itH_0} u, \quad u \in \mathcal{H} \tag{8}$$

exist and are complete. $W^{\pm}_{Y,\varepsilon}$ *satisfy*

$$\lim_{\varepsilon \to 0} \| W^{\pm}_{Y,\varepsilon} u - W^{\pm}_{\alpha,Y} u \|_{\mathcal{H}} = 0, \quad u \in \mathcal{H}. \tag{9}$$

Note that Hölder's inequality implies $V_j \in L^r(\mathbb{R}^3)$ for all $1 \leq r \leq q$ under the condition (7).

Remark 1.2. (i) It is known that $W^{\pm}_{Y,\varepsilon}$ are bounded in $L^p(\mathbb{R}^3)$ for $1 < p < 3$ ([14]) and, if $\lambda_j(\varepsilon) = 1$ for all $j = 1, \ldots, N$, $\|W^{\pm}_{Y,\varepsilon}\|_{\mathbf{B}(L^p)}$ is independent of $\varepsilon > 0$ and, the proof of Theorem 1.1 shows that Theorem 1.1 holds with $\alpha = 0$. It follows by virtue of (9) that $W_{Y,\varepsilon}$ converges to $W_{\alpha=0,Y}$ weakly in L^p and $W^{\pm}_{\alpha=0,Y}$ are bounded in $L^p(\mathbb{R}^3)$ for $1 < p < 3$. Actually, the latter result is known for general $\alpha = (\alpha_1 \ldots, \alpha_N)$ but its proof is long and complicated ([5]). Wave operators satisfy the intertwining property

$$f(H_{\alpha,Y})\mathcal{H}_{ac}(H_{\alpha,Y}) = W^{\pm *}_{\alpha,Y} f(H_0) W^{\pm *}_{\alpha,Y}$$

for Borel functions f on $\mathbb{R}$ and, L^p mapping properties of $f(H_{\alpha,Y})P_{ac}(H_{\alpha,Y})$ are reduced to those for the Fourier multiplier $f(H_0)$ for a certain range of p's.

(ii) If some of $H_j = -\Delta + V_j$ have no threshold resonance, then Theorem 1.1 remains to hold if corresponding points of interactions and parameters (y_j, α_j) are removed from $H_{\alpha,Y}$.

(iii) The first statement is long known (see [2]). We shall present here a simplified proof, providing in particular details of the proof of Lemma 1.2.3 of [2] where [6] is referred to for "a tedious but straightforward calculation" by using a result from [4] and a simple matrix formula.

(iv) The existence and the completeness of wave operators $W^{\pm}_{Y,\varepsilon}$ are well known (cf. [11]).

(v) When $N = 1$ and $\alpha = 0$, (9) is proved in [5]. The theorem is a generalization for general α and $N \geq 2$.

(vi) The matrix $\Gamma_{\alpha,Y}(k)$ is non-singular for all $k \in (0,\infty)$ by virtue of the selfadjointness of $H_{\alpha,Y}$ and H_0. Indeed, if it occurred that $\det \Gamma_{\alpha,Y}(k_0) = 0$ for some $0 < k_0$, then the selfadjointness of $H_{\alpha,Y}$ and H_0 implied that $\Gamma_{\alpha,Y}(k)^{-1}$ had a simple pole at k_0 and

$$(10) \qquad 2k_0 \mathrm{Res}_{z=k_0}(\Gamma_{\alpha,Y}(z)^{-1})_{j\ell}(\mathcal{G}_z^{y_j}, v)(u, \mathcal{G}_z^{y_\ell})$$
$$= \lim_{z=k_0+i\varepsilon,\varepsilon\downarrow 0}(z^2-k_0^2)\sum_{j,\ell=1}^{N}(\Gamma_{\alpha,Y}(z)^{-1})_{j\ell}(\mathcal{G}_z^{y_j}, v)(u, \mathcal{G}_z^{y_\ell}) \neq 0$$

for some $u, v \in C_0^\infty(\mathbb{R}^3)$. However, the absence of positive eigenvalues of $H_{\alpha,Y}$ (see [2, pp. 116–117]) and the Lebesgue dominated convergence theorem imply for all $u, v \in C_0^\infty(\mathbb{R}^3)$ that

$$\lim_{z=k_0+i\varepsilon,\varepsilon\downarrow 0}(z^2-k_0^2)((H_{\alpha,Y}-z^2)^{-1}u, v)$$
$$= \lim_{z=k_0+i\varepsilon,\varepsilon\downarrow 0}\int_{\mathbb{R}}\frac{2ik_0\varepsilon-\varepsilon^2}{\mu-(k_0+i\varepsilon)^2}(E(d\mu)u, v) = (E(\{k_0^2\})u, v) = 0$$

and the likewise for $(z^2-k_0^2)((H_0-z^2)^{-1}u, v)$, where $E(d\mu)$ is the spectral projection for $H_{\alpha,Y}$, which contradict (10).

For more about point interactions we refer to the monograph [2] or the introduction of [5] and jump into the proof of Theorem 1.1 immediately. We prove (9) only for $W_{Y,\varepsilon}^{+}$ as $\overline{H}_Y(\varepsilon)$ and $H_{\alpha,Y}$ are real operators and the complex conjugation $\mathcal{C}$ changes the direction of the time which implies $W_{Y,\varepsilon}^{-} = \mathcal{C}W_{Y,\varepsilon}^{+}\mathcal{C}^{-1}$.

We write $\mathcal{H}$ for $L^2(\mathbb{R}^3)$, (u, v) for the inner product and $\|u\|$ the norm. $u \otimes v$ and $|u\rangle\langle v|$ indiscriminately denote the one dimentional operator

$$(u \otimes v)f(x) = |u\rangle\langle v|f\rangle(x) = \int_{\mathbb{R}^3} u(x)\overline{v(y)}f(y)dy.$$

Integral operators T and their integral kernels $T(x, y)$ are identified. Thus we often say that operator $T(x, y)$ satisfies such and such properties and etc. $\mathbf{B}_2(\mathcal{H})$ is the space of Hilbert-Schmidt operators in $\mathcal{H}$ and

$$\|T\|_{HS} = \left(\iint_{\mathbb{R}^3\times\mathbb{R}^3}|T(x, y)|^2 dxdy\right)^{1/2}$$

is the norm of $\mathbf{B}_2(\mathcal{H})$. $\langle x\rangle = (1 + |x|^2)^{1/2}$ and $a \le_{|\,\cdot\,|} b$ means $|a| \le |b|$. For subsets D_1 and D_2 of the complex plane $\mathbb{C}$, $D_1 \Subset D_2$ means $\overline{D_1}$ is a compact subset of the interior of D_2.

2. Scaling

For $\varepsilon > 0$, we let

$$(U_\varepsilon f)(x) = \varepsilon^{-3/2}f(x/\varepsilon).$$

This is unitary in $\mathcal{H}$ and $H_0 = \varepsilon^2 U_\varepsilon^* H_0 U_\varepsilon$. We define $H(\varepsilon)$ by

$$(11) \quad H(\varepsilon) = \varepsilon^2 U_\varepsilon^* \overline{H}_Y(\varepsilon)U_\varepsilon, \quad (\overline{H}_Y(\varepsilon)-z^2)^{-1} = \varepsilon^2 U_\varepsilon(H(\varepsilon)-\varepsilon^2 z^2)^{-1}U_\varepsilon^*.$$

Then, $H(\varepsilon)$ is written as

$$H(\varepsilon) = -\Delta + \sum_{i=1}^{N} \lambda_i(\varepsilon) V_i \left(x - \frac{y_i}{\varepsilon} \right) \equiv -\Delta + V(\varepsilon)$$

and $W^{\pm}_{Y,\varepsilon}$ are transformed as

$$W^{\pm}_{Y,\varepsilon} = \lim_{t \to \pm\infty} U_\varepsilon e^{itH(\varepsilon)/\varepsilon^2} e^{-itH_0/\varepsilon^2} U^*_\varepsilon = U_\varepsilon W^{\pm}_Y(\varepsilon) U^*_\varepsilon, \tag{12}$$

$$W^{\pm}_Y(\varepsilon) = \lim_{t \to \pm\infty} U_\varepsilon e^{itH(\varepsilon)} e^{-itH_0} U^*_\varepsilon . \tag{13}$$

We write the translation operator by $\varepsilon^{-1} y_j$ by

$$\tau_{j,\varepsilon} f(x) = f\left(x + \frac{y_j}{\varepsilon} \right), \quad j = 1, \ldots, N.$$

When $\varepsilon = 1$, we simply denote $\tau_j = \tau_{j,1}$, $j = 1, \ldots, N$. Then,

$$V_j \left(x - \frac{y_j}{\varepsilon} \right) = \tau^*_{j,\varepsilon} V_j(x) \tau_{j,\varepsilon}.$$

3. Stationary representation

The following lemma is obvious and well known:

Lemma 3.1. *The subspace* $\mathcal{D}_* = \{u \in L^2 \colon \hat{u} \in C_0^\infty(\mathbb{R}^3 \setminus \{0\})\}$ *is a dense linear subspace of* $L^2(\mathbb{R}^3)$.

It is obvious that $\|W^+_{Y,\varepsilon} u\| = \|W^+_{\alpha,Y} u\| = \|u\|$ for every $u \in \mathcal{H}$ and, for proving (9) it suffices to show that

$$\lim_{\varepsilon \to 0} (W^+_{Y,\varepsilon} u, v) = (W^+_{\alpha,Y} u, v), \quad u, v \in \mathcal{D}_*. \tag{14}$$

We express $W^+_{Y,\varepsilon}$ and $W^+_{\alpha,Y}$ via stationary formulae. We recall from [5] the following representation formula for $W^+_{\alpha,Y}$.

Lemma 3.2. *Let* $u, v \in \mathcal{D}_*$ *and let* $\Omega_{j\ell} u$ *be defined for* $j, \ell \in \{1, \ldots, N\}$ *by*

$$\frac{1}{\pi i} \int_0^\infty \left(\int_{\mathbb{R}^3} (\Gamma_{\alpha,Y}(-k)^{-1})_{j\ell}\, \mathcal{G}_{-k}(x) \big(\mathcal{G}_k(y) - \mathcal{G}_{-k}(y) \big) u(y) dy \right) k dk. \tag{15}$$

Then

$$\langle W^+_{\alpha,Y} u, v \rangle \ = \ \langle u, v \rangle + \sum_{j,\ell=1}^{N} \langle \tau^*_j \Omega_{j\ell} \tau_\ell u, v \rangle. \tag{16}$$

Note that for $u \in \mathcal{D}_*$ *the inner integral in* (15) *produces a smooth function of* $k \in \mathbb{R}$ *which vanishes outside the compact set* $\{|\xi| \colon \xi \in \operatorname{supp} \hat{u}\}$.

For describing the formula for $W^+_{Y,\varepsilon}$ corresponding to (15) and (16), we introduce some notation. $\mathcal{H}^{(N)} = \mathcal{H} \oplus \cdots \oplus \mathcal{H}$ is the N-fold direct sum of $\mathcal{H}$.

Likewise $T^{(N)} = T \oplus \cdots \oplus T$ for an operator T on $\mathcal{H}$. For $i = 1, \dots, N$ we decompose $V_i(x)$ as the product:

$$V_i(x) = a_i(x)b_i(x), \quad a_i(x) = |V_i(x)|^{1/2}, \quad b_i(x) = |V_i(x)|^{1/2}\mathrm{sign}(V_i(x)),$$

where $\mathrm{sign}\, a = \pm 1$ if $\pm a > 0$ and $\mathrm{sign}\, a = 0$ if $a = 0$. We use matrix notation for operators on $\mathcal{H}^{(N)}$. Thus, we define

$$A = \begin{pmatrix} a_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_N \end{pmatrix}, \; B = \begin{pmatrix} b_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & b_N \end{pmatrix}, \; \Lambda(\varepsilon) = \begin{pmatrix} \lambda_1(\varepsilon) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_N(\varepsilon) \end{pmatrix}.$$

Since a_j, b_j and $\lambda_j(\varepsilon)$, $j = 1, \dots, N$ are real valued, multiplications with A, B and $\Lambda(\varepsilon)$ are selfadjoint operators on $\mathcal{H}^{(N)}$. We also define the operator τ_ε by

$$\tau_\varepsilon \colon \mathcal{H} \ni f \mapsto \tau_\varepsilon f = \begin{pmatrix} \tau_{1,\varepsilon} f \\ \vdots \\ \tau_{N,\varepsilon} f \end{pmatrix} \in \mathcal{H}^{(N)}$$

so that

$$V(\varepsilon) = \sum_{j=1}^{N} \lambda_j(\varepsilon) V_j \left(x - \frac{y_j}{\varepsilon}\right) = \tau_\varepsilon^* A \Lambda(\varepsilon) B \tau_\varepsilon.$$

We write for the case $\varepsilon = 1$ simply as $\tau = \tau_1$ as previously. For $z \in \mathbb{C}$, $G_0(z)$ is the integral operator defined by

$$G_0(z)u(y) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{e^{iz|x-y|}}{|x-y|} u(y)dy.$$

It is a holomorphic function of $z \in \mathbb{C}^+$ with values in $\mathbf{B}(\mathcal{H})$ and

$$G_0(z) = (H_0 - z^2)^{-1} \quad \text{for} \quad z \in \mathbb{C}$$

and, it can be extended to various subsets of $\mathbb{C}^+$ when considered as a function with values in a space of operators between suitable function spaces. We also write

$$G_\varepsilon(z) = (H(\varepsilon) - z^2)^{-1} \text{ for } z \in \mathbb{C}^+ \setminus \{z \colon z^2 \in \sigma_p(H(\varepsilon))\}.$$

Lemma 3.3. *Let $V_1, \dots, V_N$ satisfy the assumption* (7) *and $z \in \overline{\mathbb{C}}^+$. Then*:

(1) $a_i, b_j \in L^2(\mathbb{R}^3)$, $i, j = 1, \dots, N$.

(2) $a_i G_0(z) b_j \in \mathbf{B}_2(\mathcal{H})$, $1 \le i, j \le N$.

Proof. (1) We have $a_i, b_j \in L^2(\mathbb{R}^3)$ for $V_j \in L^1(\mathbb{R}^3)$ as was remarked below Theorem 1.1.

(2) We also have $|a_j|^2 = |b_j|^2 = |V_j| \in L^{3/2}(\mathbb{R}^3)$ and $|x|^{-2} \in L^{3/2,\infty}(\mathbb{R}^3)$. It follows by the generalized Young inequality that

$$\iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|a_i(x)|^2 |b_j(y)|^2}{|x-y|^2} dxdy \le C \|V_i\|_{L^{3/2}} \|V_j\|_{L^{3/2}}.$$

Hence, $a_i G_0(z) b_j$ is of Hilbert-Schmidt type in $L^2(\mathbb{R}^3)$. □

Using this notation, we have from (16) that

$$(W^{+}_{\alpha,Y}u,v)=(u,v)+\left\langle (\Omega_{j\ell})\tau^*u,\tau^*v\right\rangle_{\mathcal{H}^{(N)}}. \tag{17}$$

The resolvent equation for $H(\varepsilon)$ may be written as

$$G_\varepsilon(z)-G_0(z)=-G_0(z)\tau_\varepsilon^*A\Lambda(\varepsilon)B\tau_\varepsilon G_\varepsilon(z)$$

and the standard argument (see e.g. [13]) yields

$$G_\varepsilon(z)=G_0(z)-G_0(z)\tau_\varepsilon^*A(1+\Lambda(\varepsilon)B\tau_\varepsilon G_0(z)\tau_\varepsilon^*A)^{-1}\Lambda(\varepsilon)B\tau_\varepsilon G_0(z). \tag{18}$$

Note that $\tau_\varepsilon R_0(z)\tau_\varepsilon^*\neq R_0(z)$ in general unless $N=1$.

Under the assumption (7) on $V_1,\dots,V_N$ the first two statements of the following lemma follow from the limiting absorption principle for the free Schrödinger operator ([1], [7], [12]) and the last from the absence of positive eigenvalues for $H(\varepsilon)$ ([10]). In what follows we often write k for z when we want emphasize that k can also be real.

Lemma 3.4. *Suppose that $V_1,\dots,V_N$ satisfy the assumption of* Theorem 1.1. *Let $0<\varepsilon\le 1$. Then*:

1. *For $u\in\mathcal{D}_*$, $\lim_{\delta\downarrow 0}\sup_{k\in\mathbb{R}}\|A\tau_\varepsilon G_0(k+i\delta)u-A\tau_\varepsilon G_0(k)u\|_{\mathcal{H}^{(N)}}=0$.*
2. $\lim_{\delta\downarrow 0}\sup_{k\in\mathbb{R}}\|\Lambda(\varepsilon)A\tau_\varepsilon(G_0(k+i\delta)-G_0(k))\tau_\varepsilon^*A\|_{\mathbf{B}(\mathcal{H}^{(N)})}=0$.
3. *Define for $k\in\overline{\mathbb{C}}^{+}=\{k\in\Im k\ge 0\}$,*

$$M_\varepsilon(k)=\Lambda(\varepsilon)B\tau_\varepsilon G_0(k)\tau_\varepsilon^*A. \tag{19}$$

Then, $M_\varepsilon(k)$ is a compact operator on $\mathcal{H}^{(N)}$ and $1+M_\varepsilon(k)$ is invertible for all $k\neq 0$. $(1+M_\varepsilon(k))^{-1}$ is a locally Hölder continuous function of $\overline{\mathbb{C}}^{+}\setminus\{0\}$ with values in $\mathbf{B}(\mathcal{H}^{(N)})$.

Statements (1) *and* (2) *remain to hold when A is replaced by B.*

The well known stationary formula for wave operators ([12]) and the resolvent equation (18) yield

$$\begin{aligned}(W^{+}_{Y}(\varepsilon)u,v)&-(u,v)\\ &=-\frac{1}{\pi i}\int_0^\infty\Big((1+M_\varepsilon(-k))^{-1}\Lambda(\varepsilon)B\tau_\varepsilon\{G_0(k)-G_0(-k)\}u,A\tau_\varepsilon G_0(k)v\Big)\,kdk.\end{aligned} \tag{20}$$

For obtaining the corresponding formula for $W^{+}_{Y,\varepsilon}$, we scale back (20) by using the identity (12) and (13). Then

$$\tau_\varepsilon U_\varepsilon^*=U_\varepsilon^*\tau,$$

and change of variable k to εk produce the first statement of the following lemma. Recall $\tau=\tau_{\varepsilon=1}$. The second formula is proven in parallel with the first by using (11).

Lemma 3.5. (1) *For $u,v\in\mathcal{D}^*$, we have*

$$(W^{+}_{Y,\varepsilon}u,v)=(u,v)-\frac{\varepsilon^2}{\pi i}\int_0^\infty kdk\,\big((1+M_\varepsilon(-\varepsilon k))^{-1}\Lambda(\varepsilon) \tag{21}$$

$$\times\ B\{G_0(k\varepsilon) - G_0(-k\varepsilon)\}^{(N)} U_\varepsilon^* \tau u, AG_0(k\varepsilon)^{(N)} U_\varepsilon^* \tau v\Big).$$

(2) *For $k \in \mathbb{C}^+$ with sufficiently large $\Im k$,*

$$(\overline{H}_Y(\varepsilon) - k^2)^{-1} = G_0(k) - \varepsilon^2 \tau^* U_\varepsilon G_0(k\varepsilon)^{(N)} A(1 + M_\varepsilon(\varepsilon k))^{-1} \times \Lambda(\varepsilon) B G_0(k\varepsilon)^{(N)} U_\varepsilon^* \tau, \tag{22}$$

where $G_0(\pm k\varepsilon)^{(N)} = G_0(\pm k\varepsilon) \oplus \cdots \oplus G_0(\pm k\varepsilon)$ is the N-fold direct sum of $G_0(\pm k\varepsilon)$.

Notice that for $u \in \mathcal{D}_*$, $\{G_0(k\varepsilon) - G_0(-k\varepsilon)\}^{(N)} U_\varepsilon^* \tau u \neq 0$ only for $R^{-1} < k < R$ for some $R > 0$ and the integral on the right of (21) is only over $[R^{-1}, R] \subset (0, \infty)$ uniformly for $0 < \varepsilon < 1$. Indeed, if $u \in \mathcal{D}_*$ and $\hat{u}(\xi) = 0$ unless $R^{-1} \leq |\xi| \leq R$ for some $R > 1$, then, since the translation τ does not change the support of $\hat{u}(\xi/\varepsilon)$, we have

$$\mathcal{F}(U_\varepsilon^* \tau u)(\xi) = \varepsilon^{-\frac{3}{2}} \mathcal{F}(\tau u)\left(\frac{\xi}{\varepsilon}\right) = 0$$

unless $R^{-1}\varepsilon \leq |\xi| \leq R\varepsilon$ and

$$\{G_0(k\varepsilon) - G_0(-k\varepsilon)\} U_\varepsilon^* \tau u = 2i\pi \delta(\xi^2 - k^2\varepsilon^2) \mathcal{F}(U_\varepsilon^* \tau u)(\xi) = 0$$

for $k > R$ or $k < R^{-1}$.

4. Limits as $\varepsilon \to 0$

We study the small $\varepsilon > 0$ behavior of the right hand sides of (21) and (22). For (21), the argument above shows that we need only consider the integral over a compact set $K \equiv [R^{-1}, R] \subset \mathbb{R}$ *which will be fixed in this section.* Splitting $\varepsilon^2 = \varepsilon \cdot \varepsilon^{1/2} \cdot \varepsilon^{1/2}$ in front of the second term on the right, we place one $\varepsilon^{1/2}$ each in front of $BG_0(\pm k\varepsilon)^{(N)} U_\varepsilon^*$ and $AG_0(\pm k\varepsilon)^{(N)} U^*$ or $U_\varepsilon G_0(k\varepsilon)^{(N)} A$ and the remaining ε in front of $(1 + M_\varepsilon(\pm\varepsilon k))^{-1}$. We begin with the following lemma. Recall the definition (3) of $\mathcal{G}_k$.

Lemma 4.1. *Suppose $a \in L^2(\mathbb{R}^3)$. Then, following statements are satisfied:*

(1) *Let $u \in \mathcal{D}_*$. Then, uniformly in $k \in K$, we have*

$$\lim_{\varepsilon \to 0} \|\varepsilon^{\frac{1}{2}} a G_0(\pm k\varepsilon) U_\varepsilon^* u - |a\rangle\langle \mathcal{G}_{\pm k}, u\rangle\|_{L^2} = 0. \tag{23}$$

(2) *Let $u \in L^2(\mathbb{R}^3)$. Then, uniformly on compacts of $k \in \mathbb{C}^+$, we have*

$$\|\varepsilon^{\frac{1}{2}} a G_0(k\varepsilon) U_\varepsilon^* u\|_{L^2} \leq C(\Im k)^{-1/2} \|a\|_{L^2} \|u\|_{L^2} \tag{24}$$

and the convergence (23) *with k in place of $\pm k$.*

(3) *Let $u \in L^2(\mathbb{R}^3)$. Then, uniformly on compacts of $k \in \mathbb{C}^+$, we have*

$$\lim_{\varepsilon \to 0} \|\varepsilon^{\frac{1}{2}} U_\varepsilon G_0(k\varepsilon) a u - |\mathcal{G}_k\rangle\langle a, u\rangle\|_{L^2} = 0. \tag{25}$$

Proof. (1) We prove the + case only. The proof for the − case is similar. We have $u \in \mathcal{S}(\mathbb{R}^3)$ and

$$\varepsilon^{\frac{1}{2}} G_0(k\varepsilon) U_\varepsilon^* u(x) = \frac{1}{4\pi} \varepsilon^2 \int_{\mathbb{R}^3} \frac{e^{ik\varepsilon|x-y|}}{|x-y|} u(\varepsilon y) dy = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{e^{ik|y|}}{|y|} u(y + \varepsilon x) dy.$$

It is then obvious for any $R > 0$ and a compact $K \subset \mathbb{R}$ that

$$\lim_{\varepsilon \to 0} \sup_{|x| \le R, k \in K} |\varepsilon^{\frac{1}{2}} G_0(k\varepsilon) U_\varepsilon^* u(x) - \langle \mathcal{G}_k, u \rangle| = 0. \tag{26}$$

Moreover, Hölder's inequality in Lorentz spaces implies that

$$|\langle \mathcal{G}_k, u \rangle| + \|\varepsilon^{\frac{1}{2}} G_0(k\varepsilon) U_\varepsilon^* u\|_\infty \le \|(4\pi|x|)^{-1}\|_{3,\infty} \|u\|_{\frac{3}{2},1}. \tag{27}$$

It follows from (26) that for any $R > 0$

$$\lim_{\varepsilon \to 0} \sup_{k \in K} \|\varepsilon^{\frac{1}{2}} a G_0(k\varepsilon) U_\varepsilon^* u - a\langle \mathcal{G}_k, u \rangle\|_{L^2(|x| \le R)} = 0 \tag{28}$$

and, from (27) that

$$\begin{aligned} &\|\varepsilon^{\frac{1}{2}} a G_0(k\varepsilon) U_\varepsilon^* u - a\langle \mathcal{G}_k, u \rangle\|_{L^2(|x| \ge R)} \\ &\le 2\|a\|_{L^2(|x| \ge R)} \|(4\pi|x|)^{-1}\|_{3,\infty} \|u\|_{\frac{3}{2},1} \to 0. \end{aligned} \tag{29}$$

Combining (26) and (29), we obtain (23) for $u \in \mathcal{D}_*$. (Since $\mathcal{D}_*$ is dense in $L^{3,1}(\mathbb{R}^3)$, (23) actually holds for $u \in L^{\frac{3}{2},1}(\mathbb{R}^3)$.)

(2) We have

$$\|aG_0(k\varepsilon)\|_{HS}^2 = \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|a(x)|^2 e^{-2\Im k\varepsilon|x-y|}}{16|x-y|^2} dxdy \le C(\Im k\varepsilon)^{-1} \|a\|_{L^2}^2.$$

This implies (24) as U_ε^* is unitary in $L^2(\mathbb{R}^3)$ and it suffices to prove the strong convergence in L^2 for $u \in C_0^\infty(\mathbb{R}^3)$. This, however, follows as in the case (1).

(3) We have

$$\varepsilon^{\frac{1}{2}} (U_\varepsilon G_0(k\varepsilon) a u)(x) = \int_{\mathbb{R}^3} \frac{e^{ik|x-\varepsilon y|}}{4\pi|x - \varepsilon y|} a(y) u(y) dy$$

and Minkowski's inequality implies

$$\|\varepsilon^{\frac{1}{2}} U_\varepsilon G_0(k\varepsilon) a u - |\mathcal{G}_k\rangle \langle a, u \rangle\| \le \int_{\mathbb{R}^3} \|\mathcal{G}_k(\cdot - \varepsilon y) - \mathcal{G}_k\|_{L^2(\mathbb{R}^3)} |a(y) u(y)| dy. \tag{30}$$

Plancherel's and Lebesgue's dominated convergence theorems imply that for a compact subset $\tilde{K}$ of $\mathbb{C}^+$

$$\begin{aligned} \sup_{k \in \tilde{K}} \|\mathcal{G}_k(\cdot + \varepsilon y) - \mathcal{G}_k\| &= \sup_{k \in \tilde{K}} \|(\mathcal{F}^{-1}\mathcal{G}_k)(\xi)(e^{\varepsilon y \xi} - 1)\|_{L^2(\mathbb{R}^3_\xi)} \\ &= \left(\int_{\mathbb{R}^3} \sup_{k \in \tilde{K}} |(|\xi|^2 - k^2)^{-1} (e^{i\varepsilon y \xi} - 1)|^2 d\xi \right)^{\frac{1}{2}} \\ &\le C \left(\int_{\mathbb{R}^3} \langle \xi \rangle^{-4} |(e^{i\varepsilon y \xi} - 1)|^2 d\xi \right)^{\frac{1}{2}} \end{aligned}$$

is uniformly bounded for $y \in \mathbb{R}^3$ and converges to 0 as $\varepsilon \to 0$. Thus, (25) follows from (30) by applying Lebesgue's dominated convergence theorem. □

We next study $\varepsilon(1+M_\varepsilon(\varepsilon k))^{-1}$ for $\varepsilon \to 0$ and $k \in \overline{\mathbb{C}}^+ \setminus \{0\}$. We decompose $M_\varepsilon(k) = \Lambda(\varepsilon) B \tau_\varepsilon G_0(\varepsilon k) \tau_\varepsilon^* A$ into the diagonal and the off-diagonal parts:

$$M_\varepsilon(k) = D_\varepsilon(\varepsilon k) + \varepsilon E_\varepsilon(\varepsilon k), \tag{31}$$

where the diagonal part is given by

$$D_\varepsilon(\varepsilon k) = \begin{pmatrix} \lambda_1(\varepsilon) b_1 G_0(\varepsilon k) a_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_N(\varepsilon) b_N G_0(\varepsilon k) a_N \end{pmatrix} \tag{32}$$

and, the off diagonal part $\varepsilon E_\varepsilon(\varepsilon k) = \left(\lambda_i(\varepsilon) b_i \tau_{i,\varepsilon} G_0(\varepsilon k) \tau_{j,\varepsilon}^* a_j \hat{\delta}_{ij}\right)$ by

$$\varepsilon E_\varepsilon(\varepsilon k) = \varepsilon \left(\lambda_i(\varepsilon) \frac{b_i(x) e^{ik|\varepsilon(x-y)+y_i-y_j|} a_j(y)}{4\pi|\varepsilon(x-y)+y_i-y_j|} \hat{\delta}_{ij}\right)_{ij}. \tag{33}$$

We study $E_\varepsilon(\varepsilon k)$ first. Define constant matrix $\hat{\mathcal{G}}(k)$ by

$$\hat{\mathcal{G}}_{ij}(k) = \mathcal{G}_{ij}(k)\hat{\delta}_{ij}, \quad \mathcal{G}_{ij}(k) = \frac{1}{4\pi} \frac{e^{ik|y_i-y_j|}}{|y_i-y_j|}, \quad i \neq j.$$

Lemma 4.2. *Assume* (7) *and let* $\Omega \subset \overline{\mathbb{C}}^+$ *be compact. We have uniformly for* $k \in \Omega$ *that*

$$\lim_{\varepsilon \to 0} \|E_\varepsilon(\pm \varepsilon k) - |B\rangle \hat{\mathcal{G}}(\pm k) \langle A|\|_{\mathbf{B}(\mathcal{H}^{(N)})} = 0. \tag{34}$$

$|B\rangle \hat{\mathcal{G}}(\pm k) \langle A|$ *is an operator of rank at most* N *on* $\mathcal{H}^{(N)}$:

$$|B\rangle \hat{\mathcal{G}}(\pm k) \langle A| \equiv \left(b_i(x) \mathcal{G}_{ij}(\pm k) a_j(y) \hat{\delta}_{ij}\right).$$

Proof. We prove the + case only. The − case may be proved similarly. Let $k \in K$. Then,

$$\begin{aligned} &\left| \frac{e^{ik|\varepsilon(x-y)+y_i-y_j|}}{|\varepsilon(x-y)+y_i-y_j|} - \frac{e^{ik|y_i-y_j|}}{|y_i-y_j|} \right| \\ &\leq \frac{|k||\varepsilon(x-y)|}{|\varepsilon(x-y)+y_i-y_j|} + \frac{|\varepsilon(x-y)|}{|\varepsilon(x-y)+y_i-y_j||y_i-y_j|} \qquad (35) \\ &\leq \frac{C|x-y|}{|(x-y)+(y_i-y_j)/\varepsilon|} \qquad (36) \end{aligned}$$

for a constant $C > 0$ and we may estimate as

$$\begin{aligned} \|(E_{\varepsilon,ij}(\varepsilon k) - \lambda_i(\varepsilon) b_i \mathcal{G}_{ij}(k) a_j) u\|_{L^2} &\leq C \left\| \int_{\mathbb{R}^3} \frac{|b_i(x)|x-y|a_j(y)u(y)|}{|(x-y)+(y_i-y_j)/\varepsilon|} dy \right\| \\ &\leq C \left\| \int_{\mathbb{R}^3} \frac{|\langle x\rangle b_i(x) \langle y\rangle a_j(y) u(y)|}{|(x-y)+(y_i-y_j)/\varepsilon|} dy \right\| \end{aligned}$$

$$= C \left\| \int_{\mathbb{R}^3} \frac{|\tau_{i,\varepsilon}(\langle x\rangle b_i)(x)\tau_{j,\varepsilon}(\langle y\rangle a_j u)(y)|}{|x-y|} dy \right\|.$$

Since the convolution with the Newton potential $|x|^{-1}$ maps $L^{\frac{6}{5}}(\mathbb{R}^3)$ to $L^6(\mathbb{R}^3)$ by virtue of Hardy-Littlewood-Sobolev's inequality, Hölder's inequality implies that the right hand side is bounded by

$$\begin{aligned} (37) \qquad & C\|\langle x\rangle b_i\|_{L^3}\|\langle y\rangle a_j u\|_{L^{6/5}} \\ & \le C\|\langle x\rangle b_i\|_{L^3}\|\langle x\rangle a_j\|_{L^3}\|u\|_{L^2} = C\|\langle x\rangle^2 V_i\|^{\frac{1}{2}}_{L^{\frac{3}{2}}}\|\langle x\rangle^2 V_j\|^{\frac{1}{2}}_{L^{\frac{3}{2}}}\|u\|_{L^2}. \end{aligned}$$

Let $B_R(0) = \{x\colon |x| \le R\}$ for an $R > 0$. Then, for $\varepsilon > 0$ such that $4R\varepsilon < \min|y_i - y_j|$, we have

$$(35) \le 4C\varepsilon, \quad \forall x, y \in B_R(0).$$

Thus, if $V_j \in C_0^\infty(\mathbb{R}^3)$, $j = 1, \dots, N$ are supported by $B_R(0)$, then

$$\|E_\varepsilon(\varepsilon k) - \Lambda(\varepsilon)B\hat{\mathcal{G}}(k)A\|_{\mathbf{B}(\mathcal{H}^{(N)})} \le 4C\varepsilon\sum_{j=1}^N \|V_j\|_{L^1} \xrightarrow{\varepsilon\to 0} 0.$$

Since $C_0^\infty(\mathbb{R}^3)$ is a dense subspace of the Banach space $(\langle x\rangle^{-2}L^{3/2}(\mathbb{R}^3)) \cap L^1(\mathbb{R}^3)$, (37) implies $\|E_\varepsilon(\varepsilon k) - \Lambda(\varepsilon)B\hat{\mathcal{G}}(k)A\|_{\mathbf{B}(\mathcal{H}^{(N)})} \to 0$ as $\varepsilon \to 0$ for general V_j's which satisfies the assumption (7). The lemma follows because $\Lambda(\varepsilon)$ converges to the identity matrix. □

We have shown in Lemma 3.3 that $b_iG_0(k\varepsilon)a_j$ is of Hilbert-Schmidt type for $k \in \overline{\mathbb{C}}^+$ and it is well known that $1 + \lambda_j(\varepsilon)b_jG_0(k\varepsilon)a_j$ is an isomorphism of $\mathcal{H}$ unless $k^2\varepsilon^2$ is an eigenvalue of $H_j(\varepsilon) = -\Delta + \lambda_j(\varepsilon)V_j$ (see [7]). Hence, the absence of positive eigenvalues for $H_j(\varepsilon)$ (see e.g. [10]) implies that $1 + \lambda_j(\varepsilon)b_jG_0(k\varepsilon)a_j$ is an isomorphism in $\mathcal{H}$ for all $k \in \overline{\mathbb{C}}^+ \setminus (\varepsilon^{-1}i\mathcal{E}_j(\varepsilon) \cup \{0\})$ where $\mathcal{E}_j(\varepsilon) = \{k > 0\colon -k^2 \in \sigma_p(H_j(\varepsilon))\}$. Thus, if we fix a compact set $\Omega \subset \overline{\mathbb{C}}^+ \setminus \{0\}$. $1 + D_\varepsilon(\varepsilon k)$ is invertible in $\mathbf{B}(\mathcal{H}^{(N)})$ for small $\varepsilon > 0$ and $k \in \Omega$ and

$$1 + M_\varepsilon(\varepsilon k) = (1 + D_\varepsilon(\varepsilon k))(1 + \varepsilon(1 + D_\varepsilon(\varepsilon k))^{-1}E_\varepsilon(\varepsilon k)).$$

It follows that

$$(38) \qquad (1 + M_\varepsilon(\varepsilon k))^{-1} = (1 + \varepsilon(1 + D_\varepsilon(\varepsilon k))^{-1}E_\varepsilon(\varepsilon k))^{-1}(1 + D_\varepsilon(\varepsilon k))^{-1}$$

and we need study the right hand side of (38) as $\varepsilon \to 0$.

We begin by studying $\varepsilon(1 + D_\varepsilon(\varepsilon k))^{-1}$ and, since $1 + D_\varepsilon(\varepsilon k)$ is diagonal, we may do it component-wise. We first study the case $N = 1$.

4.1. Threshold analysis for the case $N = 1$

When $N = 1$, we have $M_\varepsilon(\varepsilon k) = D_\varepsilon(\varepsilon k)$.

Lemma 4.3. *Let $N = 1$, $a = a_1$ and etc. and, let Ω be compact in $\overline{\mathbb{C}}^+ \setminus \{0\}$. Then, for any $0 < \rho < \rho_0$, $\rho_0 = (3-p)/2p > 1/2$, we have following expansions in Ω in the space of Hilbert-Schmidt operators $\mathbf{B}_2(\mathcal{H})$:*

$$bG_0(k\varepsilon)a = bD_0a + ik\varepsilon bD_1a + O((k\varepsilon)^{1+\rho}), \tag{39}$$

$$M_\varepsilon(\varepsilon k) = bD_0a + \varepsilon\big(\lambda'(0)bD_0a + ikbD_1a\big) + O(\varepsilon^{1+\rho}), \tag{40}$$

$$D_0 = \frac{1}{4\pi|x-y|}, \quad D_1 = \frac{1}{4\pi}, \tag{41}$$

where $O((k\varepsilon)^{1+\rho})$ and $O(\varepsilon^{1+\rho})$ are $\mathbf{B}_2(\mathcal{H})$-valued functions of (k, ε) such that

$$\|O((k\varepsilon)^{1+\rho})\|_{HS} \le C|k\varepsilon|^{1+\rho}, \quad \|O(\varepsilon^{1+\rho})\|_{HS} \le C|\varepsilon|^{1+\rho}, \quad 0 < \varepsilon < 1, \ k \in \Omega.$$

Proof. Since $\Im k \ge 0$ for $k \in \Omega$, Taylor's formula and the interpolation imply that for any $0 \le \rho \le 1$ there exists a constant $C_\rho > 0$ such that

$$|e^{ik\varepsilon|x-y|} - (1 + ik\varepsilon|x-y|)\,| \le C_\rho|\varepsilon k|^{1+\rho}|x-y|^{1+\rho}.$$

Hence

$$\left|D_\varepsilon(\varepsilon k)(x,y) - \frac{b(x)a(y)}{4\pi|x-y|} - ik\varepsilon\frac{b(x)a(y)}{4\pi}\right| \le C_\rho|k|^{1+\rho}\varepsilon^{1+\rho}|x-y|^\rho|b(x)a(y)|.$$

We have shown in Lemma 3.3 that $D_\varepsilon(\varepsilon k)$ and bD_0a are Hilbert-Schmidt operators and bD_1a is evidently so as $a, b \in L^2(\mathbb{R}^3)$ (see the remark below Theorem 1.1). As $\langle x\rangle b(x), \langle y\rangle a(y) \in L^{2p}(\mathbb{R}^3)$, we have $\langle x\rangle^\rho a(x), \langle x\rangle^\rho a(y) \in L^2(\mathbb{R}^3)$ for $\rho < \rho_0$, and

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3} |x-y|^{2\rho}|b(x)a(y)|^2 dxdy \le C\|\langle x\rangle^\rho b(x)\|_{L^2}^2\|\langle y\rangle^\rho a(y)\|_{L^2}^2.$$

This prove estimate (39). (40) follows from (39) and Taylor's expansion of $\lambda(\varepsilon)$. This completes the proof of the lemma. □

We define

$$Q_0 = 1 + bD_0a, \quad Q_1 = \lambda'(0)bD_0a + ikbD_1a, \quad bD_1a = (4\pi)^{-1}|b\rangle\langle a|. \tag{42}$$

Regular case.

Definition. $H = -\Delta + V(x)$ is said to be of regular type at 0 if Q_0 is invertible in $\mathcal{H}$. It is of exceptional type if otherwise.

Lemma 4.4. *Suppose $N = 1$ and that $H = -\Delta + V(x)$ is of regular type at 0. Let Ω be a compact subset of $\overline{\mathbb{C}}^+$. Then*

$$\lim_{\varepsilon\to 0}\sup_{k\in\Omega}\|\varepsilon(1 + M_\varepsilon(\varepsilon k))^{-1}\|_{\mathbf{B}(\mathcal{H})} = 0. \tag{43}$$

Proof. Since $Q_0 = 1 + bD_0a$ is invertible, (40) implies the same for $1 + M_\varepsilon(\varepsilon k)$ for $k \in \Omega$ and small $\varepsilon > 0$ and,

$$\lim_{\varepsilon\to 0}\sup_{k\in\Omega}\|(1 + M_\varepsilon(\varepsilon k))^{-1} - Q_0^{-1}\|_{\mathbf{B}(\mathcal{H})} = 0.$$

(43) follows evidently. □

An application of Lemma 3.4, Lemma 4.1 and Lemma 4.4 to (21) and (22) immediately produces the following proposition for the case $N = 1$.

Proposition 4.5. *Suppose $H = -\Delta + V$ is of regular type at 0. Then*:

(1) *As $\varepsilon \to 0$, $W^+_{Y,\varepsilon}$ converges strongly to the identity operator.*

(2) *Let $\Omega_0 \subset \overline{\mathbb{C}}^+$ be compact. Then, $a(\overline{H}_Y(\varepsilon) - k^2)^{-1}b - aG_0(k)b \to 0$ in the norm of $\mathbf{B}(\mathcal{H})$ as $\varepsilon \to 0$ uniformly with respect to $k \in \Omega_0$.*

(3) *Let $\Omega_1 \Subset \mathbb{C}^+$. Then, $\lim_{\varepsilon \to 0} \sup_{k \in \Omega_1} \|(\overline{H}_Y(\varepsilon) - k^2)^{-1} - G_0(k)\|_{\mathbf{B}(\mathcal{H})} = 0$.*

Exceptional case. Suppose next that Q_0 is *not* invertible and define

$$\mathcal{M} =: \operatorname{Ker} Q_0, \quad \mathcal{N} = \operatorname{Ker} Q_0^*, \quad Q_0^* = 1 + aD_0 b.$$

By virtue of the Riesz-Schauder theorem $\dim \mathcal{M} = \dim \mathcal{N}$ are finite and $\mathcal{M}$ and $\mathcal{N}$ are dual spaces of each other with respect to the inner product of $\mathcal{H}$. Let S be the Riesz projection onto $\mathcal{M}$.

Lemma 4.6. (1) *aD_0a is an isomorphism from $\mathcal{M}$ onto $\mathcal{N}$ and bD_0b from $\mathcal{N}$ onto $\mathcal{M}$. They are inverses of each other.*

(2) *$(a\varphi, D_0a\varphi)$ is an inner product on $\mathcal{M}$ and $(b\psi, D_0b\psi)$ on $\mathcal{N}$.*

(3) *For an orthonormal basis $\{\varphi_1, \dots, \varphi_n\}$ of $\mathcal{M}$ with respect to the inner product $(a\varphi, D_0a\varphi)$, define $\psi_j = aD_0a\varphi_j$, $j = 1, \dots, n$. Then*:

  (a) *$\{\psi_1, \dots, \psi_n\}$ is an orthonormal basis of $\mathcal{N}$ with respect to $(b\psi, D_0b\psi)$.*

  (b) *$\{\varphi_1, \dots, \varphi_n\}$ and $\{\psi_1, \dots, \psi_n\}$ are dual basis of $\mathcal{M}$ and $\mathcal{N}$ respectively.*

  (c) *$Sf = \langle f, \psi_1\rangle\varphi_1 + \cdots + \langle f, \psi_n\rangle\varphi_n$, $f \in \mathcal{H}$.*

Proof. (1) Let $\varphi \in \mathcal{M}$. Then, $\varphi = -bD_0a\varphi$ and $aD_0a\varphi = -aD_0b \cdot aD_0a\varphi$. Hence $aD_0a\varphi \in \mathcal{N}$. Likewise bD_0b maps $\mathcal{N}$ into $\mathcal{M}$. We have

$$bD_0b \cdot aD_0a\varphi = (bD_0a)^2\varphi = \varphi, \quad \varphi \in \mathcal{M},$$
$$aD_0a \cdot bD_0b\psi = (aD_0b)^2\psi = \psi, \quad \psi \in \mathcal{N}$$

and aD_0a and bD_0b are inverses of each other.

(2) Let $\varphi \in \mathcal{M}$. Then $a\varphi \in L^1 \cap L^\sigma$ for some $\sigma > 3/2$ (see the proof of Lemma 4.8 below) and $\widehat{a\varphi} \in L^\infty \cap L^\rho$ for some $\rho < 3$ by Hausdorff-Young's inequality. It follows that

$$(a\varphi, D_0a\varphi) = \int_{\mathbb{R}^3} \frac{|\widehat{a\varphi}(\xi)|^2}{|\xi|^2} d\xi \geq 0$$

and $(a\varphi, D_0a\varphi) = 0$ implies $a\varphi = 0$ hence, $\varphi = -bD_0a\varphi = 0$. Thus, $(a\varphi, D_0a\varphi)$ is an inner product of $\mathcal{M}$. The proof for $(b\psi, D_0b\psi)$ is similar.

(3) We have for any $j, k = 1, \dots, n$ that

$$(b\psi_j, D_0b\psi_k) = (baD_0a\varphi_j, D_0baD_0a\varphi_k) = (-a\varphi_j, -D_0a\varphi_k) = \delta_{jk}$$

and $\{\psi_1, \ldots, \psi_n\}$ is orthonormal with respect to the inner product $(b\psi, D_0 b\psi)$. Since $n = \dim \mathcal{N}$, it is a basis of $\mathcal{N}$.

$$(\varphi_j, \psi_k) = (\varphi_j, aD_0 a\varphi_k) = (a\varphi_j, D_0 a\varphi_k) = \delta_{jk}, \quad j, k = 1, \ldots, n.$$

Hence $\{\varphi_j\}$ and $\{\psi_k\}$ are dual basis of each other. Because of this, (c) is a well known fact for Riesz projections to eigen-spaces of compact operators ([9]). This completes the proof of the lemma. □

The following lemma should be known for a long time. We give a proof for readers' convenience.

Lemma 4.7. *Let $1 < \gamma \le 2$ and $\sigma < 3/2 < \rho$. Then, the integral operator*

$$(\mathcal{Q}_\gamma u)(x) = \int_{\mathbb{R}^3} \frac{\langle y \rangle^{-\gamma} u(y)}{|x-y|} dy \tag{44}$$

is bounded from $(L^\sigma \cap L^\rho)(\mathbb{R}^3)$ to the space $C_*(\mathbb{R}^3)$ of bounded continuous functions on $\mathbb{R}^3$ which converge to 0 as $|x| \to 0$:

$$\|\mathcal{Q}_\gamma u\|_{L^\infty} \le C\|u\|_{(L^\sigma \cap L^\rho)(\mathbb{R}^3)}. \tag{45}$$

For $R \ge 1$, there exists a constant C independent of u such that for $|x| \ge R$

$$\left|(Q_\gamma u)(x) - \frac{C(u)}{|x|}\right| \le C\frac{\|u\|_{L^\sigma \cap L^\rho}}{\langle x \rangle^\gamma}, \quad C(u) = \int_{\mathbb{R}^3} \langle y \rangle^{-\gamma} u(y) dy. \tag{46}$$

Proof. We omit the index γ in the proof. Since $|x|^{-1} \in L^{3,\infty}(\mathbb{R}^3)$, it is obvious that $\mathcal{Q}u(x)$ is a bounded continuous function and that (45) is satisfied. Thus, it suffices to prove (46) for $|x| \ge 100$. Let K_x be the unit cube with center x. Combining the two integrals on the left hand side of (46), we write it as

$$\begin{aligned}(Q_\gamma u)(x) - \frac{C(u)}{|x|} &= \frac{1}{|x|}\left(\int_{K_x} + \int_{\mathbb{R}^3 \setminus K_x}\right) \frac{(2yx - y^2)\langle y \rangle^{-\gamma} u(y)}{|x-y|(|x-y| + |x|)} dy \\ &\equiv I_0(x) + I_1(x).\end{aligned}$$

When $|x - y| \le 1$ and $|x| \ge 100$, $|x|, \langle x \rangle, |y|$ and $|x - y|$ are comparable in the sense that $0 < C_1 \le |x|/\langle x \rangle \le C_2 < \infty$ and etc. and we may estimate the integral over K_x as follows:

$$|I_0(x)| \le \frac{C}{|x|\langle x \rangle^{\gamma-1}} \int_{K_x} \frac{|u(y)|}{|x-y|} dy \le \frac{C}{\langle x \rangle^\gamma}\|u\|_{L^\rho(K_x)}. \tag{47}$$

We estimate the integral $I_1(x)$ by splitting it as $I_1(x) = I_{10}(x) + I_{11}(x)$:

$$I_{10}(x) = \frac{-1}{|x|}\int_{\mathbb{R}^3 \setminus K_x} \frac{y^2 \langle y \rangle^{-\gamma} u(y)}{|x-y|(|x-y| + |x|)} dy,$$

$$I_{11}(x) = \frac{1}{|x|}\int_{\mathbb{R}^3 \setminus K_x} \frac{2yx \langle y \rangle^{-\gamma} u(y)}{|x-y|(|x-y| + |x|)} dy.$$

Since $|x-y|+|x| \geq C\langle x\rangle^{\gamma-1}\langle y\rangle^{2-\gamma}$ for $|x| \geq 100$, Hölder's inequality implies

$$|I_{10}(x)| \leq \frac{C}{|x|\langle x\rangle^{\gamma-1}} \int_{\mathbb{R}^3\setminus K_x} \frac{|u(y)|}{|x-y|} dy \leq \frac{C}{\langle x\rangle^{\gamma}} \|u\|_{L^\rho(\mathbb{R}^3)}. \tag{48}$$

Let σ' be the dual exponent of σ. Then, $\sigma' > 3$ and via Hölder's inequality

$$|I_{11}(x)| \leq C \left(\int_{\mathbb{R}^3} \left(\frac{\langle y\rangle^{1-\gamma}}{\langle x-y\rangle(\langle x\rangle + \langle y\rangle)} \right)^{\sigma'} dy \right)^{1/\sigma'} \|u\|_{L^\sigma(\mathbb{R}^3)}. \tag{49}$$

If $|x| < 100|y|$, then $\langle y\rangle^{\gamma-1}(\langle x\rangle + \langle y\rangle) \geq C\langle x\rangle^{\gamma}$ and

$$\left(\int_{|x|<100|y|} \left(\frac{\langle y\rangle^{1-\gamma}}{\langle x-y\rangle(\langle x\rangle + \langle y\rangle)} \right)^{\sigma'} dy \right)^{1/\sigma'} \leq \frac{C}{\langle x\rangle^{\gamma}} \|\langle x\rangle^{-1}\|_{L^{\sigma'}}. \tag{50}$$

When $|x| > 100|y|$, we may estimate for $1 < \gamma \leq 2$ as

$$\frac{\langle y\rangle^{1-\gamma}}{\langle x-y\rangle(|x|+|y|)} \leq \frac{C}{\langle x-y\rangle\langle x\rangle^{\gamma}}.$$

It follows that

$$\left(\int_{|x|>100|y|} \left(\frac{\langle y\rangle^{1-\gamma}}{\langle x-y\rangle(\langle x\rangle + \langle y\rangle)} \right)^{\sigma'} dy \right)^{1/\sigma'} \leq \frac{C}{\langle x\rangle^{\gamma}} \|\langle x\rangle^{-1}\|_{L^{\sigma'}}. \tag{51}$$

Estimates (50) and (51) imply

$$|I_{11}(x)| \leq \frac{C}{\langle x\rangle^{\gamma}} \|u\|_{L^\sigma}. \tag{52}$$

Combining (52) with (48), we obtain (46). □

Lemma 4.8. (1) *The following is a continuous functional on* $\mathcal{N}$:

$$\mathcal{N} \ni \varphi \mapsto L(\varphi) = \frac{1}{4\pi} \int_{\mathbb{R}^3} a(x)\varphi(x)dx = \frac{1}{4\pi}\langle a, \varphi\rangle \in \mathbb{C}.$$

(2) *For* $\varphi \in \mathcal{N}$, *let* $u = D_0(a\varphi)$. *Then,*

(a) u *is a sum* $u = u_1 + u_2$ *of* $u_1 \in C^\infty(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ *and* $u_2 \in (W^{\frac{3}{2}+\varepsilon,2} \cap W^{2,\frac{3}{2}+\varepsilon})(\mathbb{R}^3)$ *for some* $\varepsilon > 0$. *It satisfies*

$$(-\Delta + V)u(x) = 0. \tag{53}$$

(b) u *is bounded continuous and satisfies*

$$u(x) = \frac{L(\varphi)}{|x|} + O\left(\frac{1}{|x|^2}\right), \quad |x| \to \infty. \tag{54}$$

(c) u *is an eigenfunction of* H *with eigenvalue* 0 *if and only if* $L(\varphi) = 0$ *and it is a threshold resonance of* H *otherwise.*

(3) *The space of zero eigenfunctions in* $\mathcal{N}$ *has codimension at most one.*

Proof. (1) Since $a \in L^2$, $|L(\varphi)| \le (4\pi)^{-1}\|a\|_{L^2}\|\varphi\|_{L^2}$.

(2a) Assumption (7) implies $a(x) = \langle x\rangle^{-1}\tilde a(x)$ with $\tilde a \in (L^{2p}\cap L^{2q})(\mathbb{R}^3)$ and $1 \le 2p < 3$ and $2q > 6$. It follows by Hölder's inequality that $\tilde a\varphi \in L^{\frac65-\varepsilon}\cap L^{\frac32+\varepsilon}$ for an $\varepsilon > 0$. Using the the Fourier multiplier $\chi(D)$ by $\chi \in C_0^\infty(\mathbb{R}^3)$ such that $\chi(\xi) = 1$ for $|\xi| \le 1$,

$$\chi(D)u = \frac{1}{(2\pi)^{\frac32}}\int_{\mathbb{R}^3} e^{ix\xi}\chi(\xi)\hat u(\xi)d\xi,$$

we decompose u:

$$u = u_1 + u_2, \quad u_1 = \chi(D)D_0(a\varphi), \;\; u_2 = \{(1-\chi(D))(1-\Delta)D_0\}(1-\Delta)^{-1}(a\varphi).$$

Since $a\varphi \in L^1(\mathbb{R}^3)$ it is obvious that

$$u_1(x) = \frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3} e^{ix\xi}\chi(\xi)\frac{\widehat{a\varphi}(\xi)}{|\xi|^2}d\xi \in C^\infty(\mathbb{R}^3), \quad \lim_{|x|\to\infty}\partial^\alpha u_1(x) = 0$$

for all α. Since $(1-\chi(\xi))(1+|\xi|^2)|\xi|^{-2}$ is a symbol of Hörmander class S_0, the multiplier $(1-\chi(D))(1-\Delta)D_0$ is bounded in any Sobolev space $W^{k,p}(\mathbb{R}^3)$ for $1 < p < \infty$ by Mikhlin's theorem and,

$$(1-\Delta)^{-1}(a\varphi) \in W^{2,\frac32+\varepsilon}(\mathbb{R}^3)\cap W^{\frac32+\varepsilon,2}(\mathbb{R}^3)$$

for an $\varepsilon > 0$ by the Sobolev embedding theorem. It follows that

$$u_2 \in W^{2,\frac32+\varepsilon}(\mathbb{R}^3)\cap W^{\frac32+\varepsilon,2}(\mathbb{R}^3),$$

in particular, u is bounded and Hölder continuous. If $(1 + bD_0a)\varphi = 0$, then

$$a(1+bD_0a)\varphi = (1+VD_0)a\varphi = (-\Delta+V)D_0a\varphi = 0$$

and $(-\Delta + V)u(x) = 0$.

(2b) We just proved that u is bounded and Hölder continuous. We use the notation in the proof of Lemma 4.7. We have $a\varphi = -VD_0(a\varphi)$ and

$$D_0(a\varphi)(x) = \frac{1}{4\pi}\left(\int_{K_x} + \int_{\mathbb{R}^3\setminus K_x}\right)\frac{\langle y\rangle^{-1}\tilde a(y)\varphi(y)dy}{|x-y|} = I_1(x) + I_2(x).$$

Since $\langle y\rangle$ is comparable with $\langle x\rangle$ when $|x-y| < 1$,

$$|I_1(x)| \le C\langle x\rangle^{-1}\|\tilde a\varphi\|_{L^{\frac32+\varepsilon}}\||x|^{-1}\|_{L^\tau(K_x)}, \quad \tau = \tfrac{3+2\varepsilon}{1+2\varepsilon} < 3.$$

For estimating the integral over $\mathbb{R}^3 \setminus K_x$, we use that $\tilde a\varphi \in L^{\frac65-\varepsilon}$ for some $0 < \varepsilon < 1/5$. Let $\delta = (6-5\varepsilon)/(1-5\varepsilon)$. Then, $\delta > 6$ and Hölder's inequality implies

$$|I_2(x)| \le C\|\tilde a\varphi\|_{L^{\frac65-\varepsilon}}\left(\int_{\mathbb{R}^3}\frac{dy}{\langle x-y\rangle^\delta\langle y\rangle^\delta}\right)^{\frac1\delta} \le \frac{C\|\tilde a\varphi\|_{L^{\frac65-\varepsilon}}}{\langle x\rangle}.$$

Hence, $a\varphi = -VD_0(a\varphi) \in \langle x\rangle^{-3}(L^p\cap L^q)(\mathbb{R}^3)$ and Lemma 4.7 with $\gamma = 2$ implies statement (2b).

Statements (2a) and (2b) obviously implies (2c). (3) follows from (1) and (2c). □

We distinguish following three cases:

Case (a): $\mathcal{N} \cap \mathrm{Ker}\,(L) = \{0\}$. Then, Lemma 4.8 implies $\dim \mathcal{N} = 1$, H has no zero eigenvalue and has only threshold resonances $\{u = D_0(a\varphi) \colon \varphi \in \mathcal{N}\}$.

Case (b): $\mathcal{N} = \mathrm{Ker}\,(L)$. Then, $\{u = D_0(a\varphi) \colon \varphi \in \mathcal{N}\}$ consists only of eigenfunctions of H with eigenvalue 0.

Case (c): $\{0\} \subsetneqq \mathcal{N} \cap \mathrm{Ker}\,(L) \subsetneqq \mathcal{N}$. In this case H has both zero eigenvalue and threshold resonances.

In case (c), we take an orthonormal basis $\{\varphi_1, \varphi_2, \dots, \varphi_n\}$ of $\mathcal{N}$ such that $\varphi_2, \dots, \varphi_n \in \mathrm{Ker}\,(L)$ and $\varphi_1 \in \mathrm{Ker}\,(L)^\perp$ such that $L(\varphi_1) > 0$ which uniquely determines φ_1.

We study $\varepsilon(1 + M_\varepsilon(\varepsilon k))^{-1}$, $M_\varepsilon(\varepsilon k) = \lambda_0(\varepsilon) b G_0(\varepsilon k) a$ as $\varepsilon \to 0$ by applying the following Lemma 4.9 due to Jensen and Nenciu ([8]). We consider the case (c) only. The modification for the cases (a) and (b) should be obvious.

Lemma 4.9. *Let $\mathcal{A}$ be a closed operator in a Hilbert space $\mathcal{H}$ and S a projection. Suppose $\mathcal{A} + S$ has a bounded inverse. Then, $\mathcal{A}$ has a bounded inverse if and only if*

$$\mathcal{B} = S - S(\mathcal{A} + S)^{-1} S$$

has a bounded inverse in $S\mathcal{H}$ and, in this case,

$$\mathcal{A}^{-1} = (\mathcal{A} + S)^{-1} + (\mathcal{A} + S)^{-1} S \mathcal{B}^{-1} S (\mathcal{A} + S)^{-1}. \tag{55}$$

We recall (40) and (42). We apply Lemma 4.9 to

$$\mathcal{A} = 1 + M_\varepsilon(\varepsilon k) \equiv 1 + \lambda(\varepsilon) b G_0(\varepsilon k) a. \tag{56}$$

We take as S the Riesz projection onto the kernel $\mathcal{M}$ of $Q_0 = 1 + bD_0a$. Since bD_0a is compact, $Q_0 + S$ is invertible. Hence, by virtue of (40), $\mathcal{A} + S$ is also invertible for small $\varepsilon > 0$ and the Neumann expansion formula yields,

$$\begin{aligned}
(\mathcal{A} + S)^{-1} &= (Q_0 + \varepsilon Q_1 + O(\varepsilon^2) + S)^{-1} \\
&= \Big(1 + \varepsilon (Q_0 + S)^{-1} Q_1 + O(\varepsilon^2)\Big)^{-1} (Q_0 + S)^{-1} \\
&= (Q_0 + S)^{-1} - \varepsilon (Q_0 + S)^{-1} Q_1 (Q_0 + S)^{-1} + O(\varepsilon^2).
\end{aligned} \tag{57}$$

Since $S(Q_0 + S)^{-1} = (Q_0 + S)^{-1} S = S$, the operator $\mathcal{B}$ of Lemma 4.9 corresponding to $\mathcal{A}$ of (56) becomes

$$\mathcal{B} = \varepsilon S Q_1 S + O(\varepsilon^2), \quad \sup_{k \in \Omega} \|O(\varepsilon^2)\|_{\mathbf{B}(\mathcal{H})} \le C \varepsilon^2, \tag{58}$$

where $\Omega \Subset \overline{\mathbb{C}}^+ \setminus \{0\}$. Take the dual basis $(\{\varphi_j\}, \{\psi_j\})$ of $(\mathcal{M}, \mathcal{N})$ defined in Lemma 4.6. Then, $bD_0a\varphi = -\varphi$ for $\varphi \in \mathcal{M}$, $(a, \varphi_j) = 0$ for $2 \le j \le n$ and

$(\psi_j, b) = (aD_0 a\varphi_j, b) = -(\varphi_j, a)$ imply

$$SQ_1S = S(\lambda'(0)bD_0a + ikbD_1a)S = -\lambda'(0)S - \frac{ik}{4\pi}|(a,\varphi_1)|^2(\varphi_1 \otimes \psi_1).$$

It follows from (58) that uniformly with respect to $k \in \Omega$ we have

$$\left\| \varepsilon\mathcal{B}^{-1} + \left(\lambda'(0) + i\frac{k|(a,\varphi_1)|^2}{4\pi}\right)^{-1} \varphi_1 \otimes \psi_1 + \lambda'(0)^{-1}\sum_{j=2}^{n} \varphi_j \otimes \psi_j \right\| \le C\varepsilon. \tag{59}$$

Then, since $\|(\mathcal{A}+S)^{-1}\|_{\mathbf{B}(\mathcal{H})}$ is bounded as $\varepsilon \to 0$ and $k \in \Omega$ and

$$\lim_{\varepsilon\to 0}\sup_{k\in\Omega}(\|S(\mathcal{A}+S)^{-1} - S\|_{\mathbf{B}(\mathcal{H})} + \|(\mathcal{A}+S)^{-1}S - S\|_{\mathbf{B}(\mathcal{H})} = 0,$$

(55), (57) and (59) imply the first statement of the following proposition.

Proposition 4.10. *Let* $N = 1$ *and the assumption* (7) *be satisfied. Suppose that* H *is of exceptional type at* 0 *of the case* (c). *Then, with the notation of* Lemma 4.6, *uniformly with respect to* $k \in \Omega$ *in the operator norm of* $\mathcal{H}$ *we have that*

$$\begin{aligned} &\lim_{\varepsilon\to 0}\varepsilon(1 + D_\varepsilon(\varepsilon k))^{-1} \\ &= -\left(\lambda'(0) + i\frac{k|(a,\varphi_1)|^2}{4\pi}\right)^{-1} \varphi_1 \otimes \psi_1 - \lambda'(0)^{-1}\sum_{j=2}^{n} \varphi_j \otimes \psi_j \equiv \mathcal{L} \end{aligned} \tag{60}$$

and that

$$\langle a|\,(60)\,|b\rangle = -\left(\alpha - \frac{ik}{4\pi}\right)^{-1}, \quad \alpha = -\frac{\lambda'(0)}{|(a,\varphi_1)|^2}. \tag{61}$$

The same result holds for other cases with the following changes: *For the case* (a) *replace* φ_1 *and* ψ_1 *by* φ *and* ψ *respectively which are normalized as* φ_1 *and* ψ_1 *and, for the case* (b) *set* $\varphi_1 = \psi_1 = 0$.

4.2. Proof of Theorem 1.1

Let $\mathcal{L}_j$, $j = 1, \dots, N$ be the $\mathcal{L}$ of (60) corresponding to $H_j(\varepsilon) = -\Delta + \lambda_j(\varepsilon)V_j$. Then, applying Proposition 4.10 to $H_j(\varepsilon)$, we have

$$\lim_{\varepsilon\to 0}\varepsilon(1 + D_\varepsilon(\varepsilon k))^{-1} = \oplus_{j=1}^{N}\mathcal{L}_j \equiv \tilde{\mathcal{L}}. \tag{62}$$

It follows by combining Lemma 4.2 and (62) that

$$\lim_{\varepsilon\to 0}\big(1 + \varepsilon(1 + D_\varepsilon(\varepsilon k))\big)^{-1}E_\varepsilon(\varepsilon k) = 1 + \tilde{\mathcal{L}}|B\rangle\hat{\mathcal{G}}(k)\langle A|. \tag{63}$$

We apply the following lemma due to Deift ([4]) to the right of (63).

Lemma 4.11. *Suppose that* $1 + \langle A|\tilde{\mathcal{L}}|B\rangle\hat{\mathcal{G}}(k)$ *is invertible in* $\mathbf{B}(\mathbb{C}^N)$. *Then,* $1 + \tilde{\mathcal{L}}|B\rangle\hat{\mathcal{G}}(k)\langle A|$ *is also invertible in* $\mathbf{B}(\mathcal{H}^{(N)})$ *and*

$$\langle A|(1 + \tilde{\mathcal{L}}|B\rangle\hat{\mathcal{G}}(k)\langle A|)^{-1} = (1 + \langle A|\tilde{\mathcal{L}}|B\rangle\hat{\mathcal{G}}(k))^{-1}\langle A|. \tag{64}$$

Proof. Since $a_1, \dots, a_N \in L^2(\mathbb{R}^3)$, $|A\rangle \colon \mathbb{C}^N \to \mathcal{H}^{(N)}$ and $\langle A| \colon \mathcal{H}^{(N)} \to \mathbb{C}^N$ are both bounded operators. Then, the lemma is an immediate consequence of Theorem 2 of [4]. □

For the next lemma we use the following simple lemma for matrices. Let

$$\mathcal{A} = \begin{pmatrix} W & X \\ Y & Z \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} 0 & 0 \\ 0 & V \end{pmatrix}$$

be matrices decomposed into blocks.

Lemma 4.12. *Suppose V and $1 + VZ$ are invertible. Then,*

$$\left(1 + \begin{pmatrix} 0 & 0 \\ 0 & V \end{pmatrix} \begin{pmatrix} W & X \\ Y & Z \end{pmatrix}\right)^{-1}$$

exists and

$$\left(1 + \begin{pmatrix} 0 & 0 \\ 0 & V \end{pmatrix} \begin{pmatrix} W & X \\ Y & Z \end{pmatrix}\right)^{-1} \begin{pmatrix} 0 & 0 \\ 0 & V \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & (V^{-1} + Z)^{-1} \end{pmatrix}. \tag{65}$$

Proof. It is elementary to see

$$\begin{aligned} \left(1 + \begin{pmatrix} 0 & 0 \\ 0 & V \end{pmatrix} \begin{pmatrix} W & X \\ Y & Z \end{pmatrix}\right)^{-1} &= \begin{pmatrix} 1 & 0 \\ VY & 1 + VZ \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 1 & 0 \\ -(1 + VZ)^{-1} VY & (1 + VZ)^{-1} \end{pmatrix} \end{aligned} \tag{66}$$

and the left side of (65) is equal to

$$\begin{pmatrix} 0 & 0 \\ 0 & (1 + VZ)^{-1} V \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & (V^{-1} + Z)^{-1} \end{pmatrix}$$

which proves the lemma. □

Lemma 4.13. *Let $k \in \Omega$. Then, $1 + \langle A|\tilde{\mathcal{L}}|B\rangle \hat{\mathcal{G}}(k)$ is invertibe in $\mathbb{C}^N$. If $H_1, \dots, H_N$ are arranged in such a way that $H_1, \dots, H_{n_1}$ have no resonances and $H_{n_1+1}, \dots, H_N$ do and, $N = n_1 + n_2$, then*

$$(1 + \langle A|\tilde{\mathcal{L}}|B\rangle \hat{\mathcal{G}}(k))^{-1} \langle A|\tilde{\mathcal{L}}|B\rangle = \begin{pmatrix} \mathbb{O}_{n_1 n_1} & \mathbb{O}_{n_1 n_2} \\ \mathbb{O}_{n_2 n_1} & -\tilde{\Gamma}(k)^{-1} \end{pmatrix}, \tag{67}$$

where $\mathbb{O}_{n_1 n_1}$ is the zero matrix of size $n_1 \times n_1$ and etc. and

$$\tilde{\Gamma}(k) = \left(\left(\alpha_j - \frac{ik}{4\pi}\right)\delta_{j,\ell} - \mathcal{G}_k(y_j - y_\ell)\hat{\delta}_{j\ell}\right)_{j,\ell = n_1+1,\dots,N}. \tag{68}$$

Proof. We let φ_{j1} be the resonance of H_j, $j = n_1 + 1, \dots, N$, corresponding to φ_1 of the previous section and define

$$\alpha_j = -\frac{\lambda'(0)}{|(a_j, \varphi_{j1})|^2}. \tag{69}$$

Then, Proposition 4.10 implies that,

$$\langle A|\tilde{\mathcal{L}}|B\rangle = \begin{pmatrix} 0 & & & & & \\ & \ddots & & & & \\ & & 0 & & & \\ & & & -\left(\alpha_{n_2+1} - \frac{ik}{4\pi}\right)^{-1} & & \\ & & & & \ddots & \\ & & & & & -\left(\alpha_{n_1+n_2} - \frac{ik}{4\pi}\right)^{-1} \end{pmatrix}$$

and we obtain (67) by applying Lemma 4.12 to the left of (67) with

$$V = \begin{pmatrix} -\left(\alpha_{n_2+1} - \frac{ik}{4\pi}\right)^{-1} & & \\ & \ddots & \\ & & -\left(\alpha_{n_1+n_2} - \frac{ik}{4\pi}\right)^{-1} \end{pmatrix}$$

and with

$$\begin{pmatrix} W & X \\ Y & Z \end{pmatrix} = \hat{\mathcal{G}}(k).$$

□

Lemma 4.11 and Lemma 4.13 imply that the following limit exists in $\mathbf{B}(\mathcal{H})$ and

$$\lim_{\varepsilon\to 0} \big(1 + \varepsilon(1 + D_\varepsilon(\varepsilon k))^{-1} E_\varepsilon(\varepsilon k)\big)^{-1} = \big(1 + \tilde{\mathcal{L}}|B\rangle\hat{\mathcal{G}}(k)\langle A|\big)^{-1}$$

and hence so does

$$\lim_{\varepsilon\to 0} \varepsilon\big(1 + M_\varepsilon(\varepsilon k)\big)^{-1} = \big(1 + \tilde{\mathcal{L}}|B\rangle\hat{\mathcal{G}}(k)\langle A|\big)^{-1}\tilde{\mathcal{L}}. \tag{70}$$

Completion of the proof of Theorem 1.1. By the assumption of the theorem, we may assume $n_1 = 0$ in Lemma 4.13. Abusing notation, we write

$$\hat{\mathcal{G}}_k^{(N)} u = (\hat{\mathcal{G}}_k u)^{(N)}, \quad \hat{\mathcal{G}}_k u = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{e^{ik|x|}u(x)}{|x|}dy.$$

We first prove (9) for the + case. We let $u, v \in \mathcal{D}_*$ and $R > 0$. Then, (23) and (70) imply that

$$\varepsilon^2((1 + M_\varepsilon(-\varepsilon k))^{-1}\Lambda(\varepsilon)B(G_0(k\varepsilon) - G_0(-k\varepsilon))^{(N)}U_\varepsilon u, AG_0(k\varepsilon)^{(N)}U_\varepsilon v) \tag{71}$$

converges as $\varepsilon \to 0$ to

$$(\langle A|(1 + \tilde{\mathcal{L}}|B\rangle\hat{\mathcal{G}}(-k)\langle A|)^{-1}\tilde{\mathcal{L}}|B\rangle\langle(\mathcal{G}_k^{(N)} - \mathcal{G}_{-k}^{(N)})u, \mathcal{G}_k^{(N)}v) \tag{72}$$

uniformly with respect to $k \in [R^{-1}, R]$. Here we have

$$\begin{aligned} \langle A|(1 + \tilde{\mathcal{L}}|B\rangle\hat{\mathcal{G}}(-k)\langle A|)^{-1}\tilde{\mathcal{L}}|B\rangle &= (1 + \langle A|\mathcal{L}|B\rangle\hat{\mathcal{G}}(-k))^{-1}\langle A|\mathcal{L}|B\rangle \\ &= -\tilde{\Gamma}(-k)^{-1} \end{aligned} \tag{73}$$

by virtue of (64) and (67). Thus, (71) converges as $\varepsilon \to 0$ to

$$-(\Gamma_{\alpha,Y}(-k)^{-1}\big(\hat{\mathcal{G}}_k - \hat{\mathcal{G}}_{-k}\big)^{(N)}u, \hat{\mathcal{G}}_k^{(N)}v)$$

uniformly on $[R^{-1}, R]$. Thus, replacing u and v respectively by τu and τv, we obtain $W^+_{Y,\varepsilon} \to W^+_{\alpha,Y}$ strongly as $\varepsilon \to 0$ in view of (15) and (21).

By virtue of (1) and (22), for proving the convergence (6) of the resolvent, it suffices to show that as $\varepsilon \to 0$ in the strong topology of $\mathbf{B}(\mathcal{H})$

$$\begin{aligned} &\varepsilon^2 U_\varepsilon G_0(k\varepsilon)^{(N)} A(1+M_\varepsilon(\varepsilon k))^{-1}\Lambda(\varepsilon)\varepsilon B G_0(k\varepsilon)^{(N)} U_\varepsilon \\ &\to -|\hat{\mathcal{G}}_k^{(N)}\rangle \Gamma_{\alpha,Y}(k)^{-1}\langle \hat{\mathcal{G}}_k^{(N)}| \end{aligned} \tag{74}$$

for every $k \in \mathbb{C}^+ \setminus \mathcal{E}$. However, (23), (25) and (70) imply that for $k \in \mathbb{C}^+ \setminus \mathcal{E}$ the first line of (74) converges strongly in $\mathbf{B}(\mathcal{H})$ as $\varepsilon \to 0$ to

$$|\mathcal{G}_k^{(N)}\rangle\langle A|(1+\tilde{\mathcal{L}}|B\rangle\hat{\mathcal{G}}(k)\langle A|)^{-1}\tilde{\mathcal{L}}|B\rangle\langle\mathcal{G}_k^{(N)}|. \tag{75}$$

This is equal to the second line by virtue of (73) with k in place of $-k$. This completes the proof of the theorem.

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МОНГОЛ УЛСЫН ИХ СУРГУУЛЬ

МАТЕМАТИКИЙН ТЭНХИМ, БУС, ШУС
ХЭРЭГЛЭЭНИЙ МАТЕМАТИКИЙН ТЭНХИМ, ХШУИС

ЕРДИЙН ДИФФЕРЕНЦИАЛ ТЭГШИТГЭЛ

MAPLE СИСТЕМЭЭР БОДОХ НЬ

Улаанбаатар хот
2020 он

Монгол Улсын Их Сургууль

ШУС, БУС, Математикийн тэнхим

ХШУИС, Хэрэглээний Математикийн тэнхим

Ердийн дифференциалт тэгшитгэл

MAPLE системээр бодох нь

Улаанбаатар хот

2020 он

Өмнөх үг

Орчин үед их, дээд сургуульд заадаг математикийн чиглэлийн хичээлийн сургалтыг оюутнуудад илүү сонирхолтой, үр дүнтэй болгох сургалтын нэг шинэлэг арга барил нь тухайн хичээлд тохирсон программ хангамжийг уламжлалт заах арга зүйтэй хослуулан хэрэглэх явдал юм. Эдүгээ АНУ, Канад, Их Британи, Герман зэрэг хөгжингүй орнуудын коллеж, их сургуулиудад Анализ, Шугаман алгебр, Дифференциалт тэгшитгэл, Тоон анализ, Дифференциалт геометр, Динамик систем, Математик статистикийн хичээлүүд төдийгүй Тооны онол, Бүлгийн онол, (Вектор талбарын) Ли алгебр гэх мэт олон хичээлийн сургалтанд Maple, Matlab, Mathematica, MathCad, R, Python зэрэг мэргэжлийн программ хангамжийг ашиглаж байна. Иймээс сүүлийн жилүүдэд МУИС-д хийгдэж байгаа сургалтын хөтөлбөрийн шинэчлэл, эх хэл дээр мэргэжлийн ном, сурах бичиг ихээхэн хомс байгаа болон сургалт, судалгаанд дэвшилтэт арга барил, техник хэрэгсэл ашиглах, нэвтрүүлэх зэрэгтэй уялдан Дифференциалт тэгшитгэл, Динамик системийн хичээл судалж буй оюутнуудад зориулж энэхүү номыг бичлээ.

Энэ номын зорилго нь математикийн хүчирхэг программ хангамж болох Maple системээр Ердийн дифференциалт тэгшитгэлийг хэрхэн бодох талаар танилцуулахад орших бөгөөд оюутнууд өөрсдөө бие даан уншиж, тухайн сэдвийн агуулгыг ойлгон эзэмшихэд дөхөм болохуйц жишээ, тайлбараар баяжуулахыг хичээлээ. Суралцагч англи хэл, программчлалын анхан түвшний мэдлэг, туршлагатай бол ямар ч хүндрэл, түвэггүйгээр бүх сэдвийг уншин ойлгож, жишээний кодыг туршин ажиллуулж чадна.

Уг ном 2 үндсэн хэсгээс тогтох бөгөөд эхний бүлгүүдэд

- Maple системийн орчинд янз бүрийн хүндрэлтэй математик тооцоолол гүйцэтгэх; өргөн хэрэглэгддэг коммандууд, өгөгдлийн төрлүүдтэй ажиллах; 2- ба 3- хэмжээст график байгуулах; хялбар процедур зохиох

- Ердийн дифференциалт тэгшитгэл болон системийн жинхэнэ шийдийг олох; чиглэлийн оронг зурах; Лапласын хувиргалтыг бодох; фазын диаграмм байгуулах; анхны утгат бодлогын аналитик ба тоон шийдийг тооцоолох

зэрэг анхан шатны элементүүдийн талаар товч өгүүлнэ. Сүүлчийн бүлэгт Maple хэлний процедур, Maple орчинд хэрэглэгчийн программ хэрхэн бичих талаар анхлан суралцагчийн түвшинд танилцах болно. Дашрамд хэлэхэд аль ч их сургуулийн бакалаврын түвшний математикийн хөтөлбөрт багтдаг Ердийн дифференциалт тэгшитгэлийн хичээлээр судалдаг олон төрлийн дифференциалт тэгшитгэлийг уламжлалт ангиллаар нь (хувьсагч нь ялгагддаг эсэх, бүтэн дифференциалт болох эсэх, хэддүгээр эрэмбийнх, нэгэн төрлийн ба нэгэн төрлийн биш эсэх, тогтмол эсвэл хувьсах коэффициенттэй, шугаман ба шугаман бус гэх мэтээр нь) ангилан авч үзээгүй юм. Учир нь эдгээр тэгшитгэлийн аналитик ба тоон шийдийг Maple системийн ганцхан коммандаар тооцоолон олдог. Иймээс энэхүү хүчирхэг, универсал комманд болох *dsolve* болон бусад өргөн хэрэглэгддэг комманд, операторуудыг аль болох дэлгэрэнгүй авч үзлээ. Түүнчлэн, Maple нь жил бүр хөгжин өөрчлөгдөж байгаа учир энэ системийг техникийн талаас нь авч үзээгүй болно.

Сонирхсон оюутнуудыг зөвхөн хичээлийн богино хугацаанд олж авсан багахан мэдлэг, мэдээллээр өөрийгөө хязгаарлахгүйгээр (интернэт ашиглан) бие дааж болон бусадтай хамтарч илүү ихэд суралцахыг зөвлөе. Уншигч та энэ номын талаарх санал, зөвлөмжөө дараах и-мэйл хаягаар ирүүлээрэй: gantulga@smcs.num.edu.mn

Энэхүү номыг хэвлүүлэхэд туслалцаа үзүүлсэн МУИС Пресс Хэвлэлийн газрын хамт олондоо зохиогчдын зүгээс гүн талархал илэрхийлье.

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A solvable model of the breakdown of the adiabatic approximation

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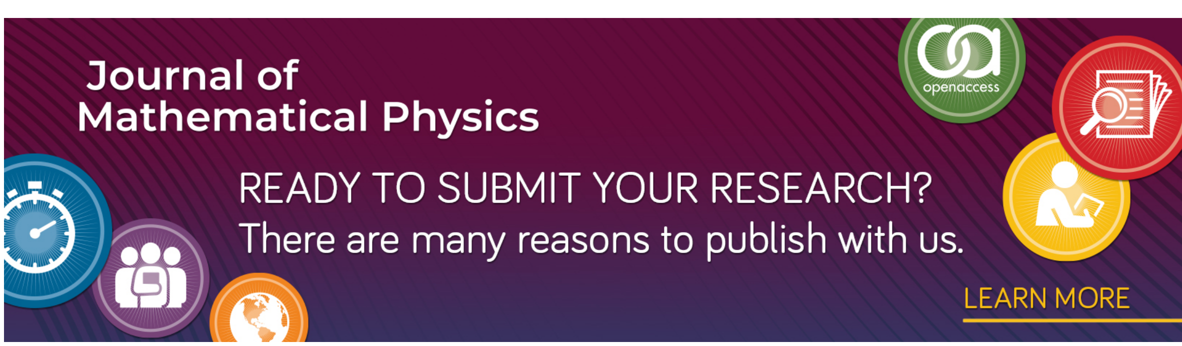

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ABSTRACT

Let $L \geq 0$ and $0 < \varepsilon \ll 1$. Consider the following time-dependent family of $1D$ Schrödinger equations with scaled harmonic oscillator potentials $i\varepsilon\partial_t u_\varepsilon = -\frac{1}{2}\partial_x^2 u_\varepsilon + V(t,x)u_\varepsilon$, $u_\varepsilon(-L-1,x) = \pi^{-1/4}\exp(-x^2/2)$, where $V(t,x) = (t+L)^2x^2/2$, $t < -L$, $V(t,x) = 0$, $-L \leq t \leq L$, and $V(t,x) = (t-L)^2x^2/2$, $t > L$. The initial value problem is explicitly solvable in terms of Bessel functions. Using the explicit solutions, we show that the adiabatic theorem breaks down as $\varepsilon \to 0$. For the case $L = 0$, complete results are obtained. The survival probability of the ground state $\pi^{-1/4}\exp(-x^2/2)$ at microscopic time $t = 1/\varepsilon$ is $1/\sqrt{2} + O(\varepsilon)$. For $L > 0$, the framework for further computations and preliminary results are given.

I. INTRODUCTION

Let $\mathcal{H}$ be a Hilbert space and $\{H(t) : -a < t < a\}$ be a family of self-adjoint operators in $\mathcal{H}$. Suppose that the time-dependent Schrödinger equation

$$i\varepsilon\partial_t u(t) = H(t)u(t) \tag{1}$$

with a small parameter $0 < \varepsilon \ll 1$ generates a unique unitary propagator $U_\varepsilon(t,s)$ and that $t \mapsto (H(t)-i)^{-1} \in \mathbf{B}(\mathcal{H})$ is of class $PC^2(\mathbb{R})$, i.e., piecewise C^2. Suppose further that $H(t)$ has an isolated simple eigenvalue $\lambda(t)$ with an associated normalized eigenfunction $\varphi(t)$, both of class PC^1 for $t \in (-a,a)$, such that

$$H(t)\varphi(t) = \lambda(t)\varphi(t), \quad \|\varphi(t)\|^2 = 1, \quad -a < t < a.$$

Then, the classical theorem of adiabatic approximation due to Born and Fock[2] and Kato[8] implies that the solution $u_\varepsilon(t) = U_\varepsilon(t,0)\varphi(0)$ of the initial value problem,

$$i\varepsilon\partial_t u_\varepsilon(t) = H(t)u_\varepsilon(t), \quad u_\varepsilon(0) = \varphi(0), \tag{2}$$

with a small parameter $0 < \varepsilon \ll 1$ satisfies for $\delta < a$

$$\|u_\varepsilon(t) - e^{-i\varepsilon^{-1}\int_0^t \lambda(s)ds}\varphi(t)\| \leq C_\delta\varepsilon, \quad |t| < \delta, \tag{3}$$

where $\|\cdot\|$ is the norm of $L^2(\mathbb{R})$. More precisely, a gap condition is imposed, i.e., assume that

$$\inf\{\text{dist}(\{\lambda(t)\}, \sigma(H(t))\backslash\{\lambda(t)\}) : -\delta < t < \delta\} > 0.$$

Furthermore, the phase of $\varphi(t)$ has to be fixed correctly. Let

$$P(t) = -\frac{1}{2\pi i}\int_{|z-\lambda(t)|=\eta}(H(t)-z)^{-1}dz$$

be the (Riesz) projection onto the eigenspace $\text{Ker}(H(t)-\lambda(t))$. Then, for a sufficiently small $\delta > 0$, one defines for $-\delta < t < \delta$

$$\varphi(t) = \frac{P(t)\varphi(0)}{\|P(t)\varphi(0)\|}.$$

With this choice, the result (3) holds.

This adiabatic theorem has been substantially elaborated and extended to more general situations, and it has been widely applied in various fields of mathematical physics; see, e.g., Teufel's monograph[10] and the references therein.

A. The eigenvalue dives into the continuum

We consider the situation that eigenvalue $\lambda(t)$ dives into the continuous spectrum of $H(t)$ at, say, $t = -L > -a$, stays in the continuum of $H(t)$ for $-L \le t \le L$, and comes out again for $t > L$ as an isolated eigenvalue of $H(t)$. Under the assumption that $\lambda(t)$ remains as an (embedded) eigenvalue of $H(t)$ for $-L \le t \le L$, then a general argument has been established and a result similar to (3) is obtained (see Ref. 10). Moreover, the result has been applied by Dürr and Pickl[5] to the Dirac equation to explain the adiabatic pair creation and by Cornean *et al.*[3] to specific finite rank perturbations of Schrödinger equations. However, if $H(t)$ has no embedded eigenvalues for $-L \le t \le L$ and the eigenvalue $\lambda(t)$ "melts away into the continuum," then there is no general theory to deal with the problem; it is even not clear what is meant by the adiabatic approximation. We should mention that embedded eigenvalues in the continuum are very unstable under a perturbation, and for genuinely time-dependent Hamiltonians, embedded eigenvalues would hardly persist for any finite time interval.

B. Harmonic oscillators that become the free Hamiltonian

To understand these phenomena, we study an explicitly solvable model. More precisely, we study the solution of the Schrödinger equation that can be written in terms of the macroscopic time variable as

$$i\varepsilon\partial_t u_\varepsilon = -\frac{1}{2}\partial_x^2 u_\varepsilon + V(t,x)u_\varepsilon, \quad u_\varepsilon(-L-1,x) = \varphi_0(x), \tag{4}$$

which is a scaled harmonic oscillator for $t < -L$ and $t > L$ and $V(t,x) = 0$ for $-L \le t \le L$,

$$V(t,x) = \begin{cases}(t+L)^2x^2/2, & t < -L,\\ 0, & -L \le t \le L,\\ (t-L)^2x^2/2, & t > L,\end{cases} \tag{5}$$

and the initial state $\varphi_0(x) = \pi^{-\frac{1}{4}}e^{-x^2/2}$ is the normalized ground state of the initial Hamiltonian $H(-L-1) = -(1/2)\partial_x^2 + (1/2)x^2$. We are particularly interested in the asymptotic behavior as $\varepsilon \to 0$ of $u_\varepsilon(t,x)$ at $t = L+1$ when $H(t)$ again becomes $-(1/2)\partial_x^2 + (1/2)x^2$.

It is well known that Eq. (4) generates a unique unitary propagator $\{U_\varepsilon(t,s) : -\infty < t, s < \infty\}$, which is simultaneously an isomorphism of $\mathcal{S}(\mathbb{R})$ and of $\Sigma(2n)$, $n = 0, 1, \ldots$, the domain of $(-(1/2)\partial_x^2 + (1/2)x^2)^n$. For $\varphi \in \Sigma(2)$, $\mathbb{R}\times\mathbb{R} \ni (t,s) \mapsto U_\varepsilon(t,s)\varphi \in L^2(\mathbb{R})$ is C^1 in (t,s) and $u_\varepsilon(t) = U_\varepsilon(t,-L-1)\varphi$ (see Ref. 6). We should emphasize, however, that $H(t)$ fails to satisfy the assumptions of the theory of adiabatic approximation in two ways: (1) all eigenvalues dive into continuum simultaneously and (2) the domain of $H(t)$ has a sharp transition at time $t = -L$ and $t = L$ and the resolvent $(H(t)-i)^{-1}$ is not of class C^1 at these points.

We shall study (4) in the microscopic time variable, viz., we change the time variable to $s = t/\varepsilon$ and study $v_\varepsilon(s,x) = u_\varepsilon(\varepsilon s, x)$. $v_\varepsilon(s,x)$ satisfies

$$i\partial_s v_\varepsilon = -\frac{1}{2}\partial_x^2 v_\varepsilon + V(\varepsilon s, x)v_\varepsilon \quad v_\varepsilon(-\varepsilon^{-1}(L+1),x) = \varphi_0(x), \tag{6}$$

and as we only consider (6) in what follows, *we denote the microscopic time variable again by t instead of s.* Our result will be rather complete in the case $L = 0$; however, when $L > 0$, the situation becomes exceedingly complicated and we have to be satisfied with partial results that should be considered as the starting point for further study.

C. Summary of results

The main results in the case $L = 0$ are stated in Theorem II.10. Let $v_\varepsilon(t, x)$ denote the solution to (6) with initial state $\varphi_0(x) = \pi^{-\frac{1}{4}} e^{-x^2/2}$ (at time $t = -1/\varepsilon$). Then, at time $t = 1/\varepsilon$, we have

$$v_\varepsilon(1/\varepsilon, x) = m_{\varepsilon,0}(1/\varepsilon) e^{-l^*_{\varepsilon,0}(1/\varepsilon)x^2/2} + O(\varepsilon)$$

as $\varepsilon \to 0$. Here, $m_{\varepsilon,0}(1/\varepsilon)$ and $l^*_{\varepsilon,0}(1/\varepsilon)$ are given by (46) and (45), respectively. Note that these coefficients are highly oscillatory as $\varepsilon \to 0$, exhibiting the breakdown of the adiabatic approximation.

This result allows one to compute the survival probability of the time $t = -1/\varepsilon$ initial state φ_0 to time $t = 1/\varepsilon$. The result is

$$|\langle v_\varepsilon(1/\varepsilon, x), \varphi_0(x)\rangle|^2 = \frac{1}{\sqrt{2}} + O(\varepsilon)$$

as $\varepsilon \to 0$. Thus, the initial state survives with a positive probability, which is less than 1. This survival probability was computed in Ref. 1 by different methods.

The results in the case $L > 0$ are stated in Theorem III.2. These partial results are somewhat complicated to state. Roughly, there exist sequences $\varepsilon_n \to 0$ as $n \to \infty$ such that there is a positive survival probability of the initial state, which, however, rapidly tends to zero as the length of (microscopic) time $2L/\varepsilon$ spent in the continuum increases.

II. THE CASE $L = 0$

We first consider the case $L = 0$, viz., the case where the eigenvalues of $H(t)$ touch upon the continuum only at time $t = 0$ but all simultaneously. We should mention that the problem for this case has been studied by Bachmann *et al.*[1] by using a method very different from ours, and the results slightly overlap.

We record a few lemmas that we shall use in what follows. The first one can be found in Ref. 11.

Lemma II.1. Let $l_\varepsilon(t)$ be the solution of the Riccati equation,

$$l'_\varepsilon(t) + i l_\varepsilon(t)^2 = i\varepsilon^2 t^2 \tag{7}$$

with initial condition

$$l_\varepsilon(-1/\varepsilon) = 1. \tag{8}$$

Suppose that $m_\varepsilon(t)$ solves

$$i m'_\varepsilon(t) = \frac{1}{2} m_\varepsilon(t) l_\varepsilon(t), \quad m_\varepsilon(-1/\varepsilon) = \pi^{-\frac{1}{4}}. \tag{9}$$

Then, $|m_\varepsilon(t)|^4 = \pi^{-1}\mathrm{Re}\, l_\varepsilon(t)$ and

$$v_\varepsilon(t, x) = m_\varepsilon(t) e^{-l_\varepsilon(t)x^2/2} \tag{10}$$

is the solution of the initial value problem for the Schrödinger equation

$$i\partial_t v_\varepsilon = -(1/2)\partial_x^2 v_\varepsilon + (t^2\varepsilon^2 x^2/2) v_\varepsilon, \quad v_\varepsilon(-1/\varepsilon, x) = \pi^{-\frac{1}{4}} e^{-\frac{x^2}{2}}. \tag{11}$$

A. General solutions of the Riccati equation

Bessel functions of the first kind $J_\nu(z)$ and the second kind $Y_\nu(z)$ are defined by

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} (-1)^k \frac{(z^2/4)^k}{k!\,\Gamma(\nu + k + 1)}, \tag{12}$$

$$Y_\nu(z) = \frac{J_\nu(z)\cos\nu\pi - J_{-\nu}(z)}{\sin\nu\pi}. \tag{13}$$

They are linearly independent solutions of Bessel's equation

$$z^2 J''_\nu(z) + z J'_\nu(z) + (z^2 - \nu^2) J_\nu(z) = 0,$$

and their positive zeros are interlaced [see DLMF[9] (10.21.3)].

Lemma II.2. Let $\varepsilon > 0$ *and* $\kappa \in (\mathbb{C} \cup \{\infty\})\backslash\mathbb{R}$. *Define* $w(s,\kappa)$ *for* $s \geq 0$ *by*

$$w(s,\kappa) = s^{1/8}\Big(-J_{-\frac{1}{4}}(2\sqrt{s}) + \kappa J_{\frac{1}{4}}(2\sqrt{s})\Big), \tag{14}$$

where the principal branches are assumed for the Bessel functions, and in the case $\kappa = \infty$, *the first term is omitted. Define*

$$\tilde{w}_\varepsilon(t,\kappa) = w\left(\frac{\varepsilon^2 t^4}{16}, \kappa\right), \quad t > 0. \tag{15}$$

Then, $\tilde{w}_\varepsilon(t,\kappa)$ *may be analytically continued to an entire function of* $t \in \mathbb{C}$ *and it does not vanish on the real line.*

Proof. From the definition of Bessel functions (12), we have

$$J_\nu(2\sqrt{s}) = s^{\nu/2} M_\nu(s), \quad M_\nu(s) = \sum_{k=0}^{\infty} (-1)^k \frac{s^k}{k!\,\Gamma(\nu+k+1)}, \tag{16}$$

and $M_\nu(s)$ is evidently an entire function of $s \in \mathbb{C}$. It follows that

$$w(s,\kappa) = -M_{-\frac{1}{4}}(s) + \kappa s^{\frac{1}{4}} M_{\frac{1}{4}}(s) \tag{17}$$

and

$$\tilde{w}_\varepsilon(t,\kappa) = -M_{-\frac{1}{4}}(\varepsilon^2 t^4/16) + (\kappa\sqrt{\varepsilon}t/2) M_{\frac{1}{4}}(\varepsilon^2 t^4/16) \tag{18}$$

is an entire function of $t \in \mathbb{C}$. As $w(s,\kappa)$ is a linear combination of $J_{\frac{1}{4}}(2\sqrt{s})$ and $J_{-\frac{1}{4}}(2\sqrt{s})$ with non $\mathbb{R}$-related coefficients, $\tilde{w}_\varepsilon(t,\kappa) \neq 0$ for $t > 0$. However, (18) shows $\tilde{w}_\varepsilon(-t,\kappa) = \tilde{w}_\varepsilon(t,-\kappa)$, and the same is true for $t < 0$, and $\tilde{w}_\varepsilon(0,\kappa) = -M_{-\frac{1}{4}}(0) = -\Gamma(3/4)^{-1} \neq 0$. This completes the proof. □

Lemma II.3. Let $\varepsilon \neq 0$, $\kappa \in (\mathbb{C} \cup \{\infty\})\backslash\mathbb{R}$, *and* $s = \varepsilon^2 t^4/16$. *Let* $w(s,\kappa)$ *and* $\tilde{w}_\varepsilon(t,\kappa)$ *be as in Lemma II.2.*

(1) *With* κ *being an arbitrary constant, the general solution of the Riccati equation* (7) *is given by*

$$l_\varepsilon(t,\kappa) = \frac{-4isw'(s,\kappa)}{tw(s,\kappa)} = -i\frac{\tilde{w}'_\varepsilon(t,\kappa)}{\tilde{w}_\varepsilon(t,\kappa)}. \tag{19}$$

It is a holomorphic function of t *in a complex neighborhood of the real line.*

(2) *We may express* $l_\varepsilon(t,\kappa)$ *without using derivatives,*

$$l_\varepsilon(t,\kappa) = \frac{-4is^{\frac{1}{2}}\Big(\kappa J_{-\frac{3}{4}}(2\sqrt{s}) + J_{\frac{3}{4}}(2\sqrt{s})\Big)}{t\Big(-J_{-\frac{1}{4}}(2\sqrt{s}) + \kappa J_{\frac{1}{4}}(2\sqrt{s})\Big)} \tag{20}$$

$$= \frac{-i\Big(8\kappa\varepsilon^{\frac{1}{2}} M_{-\frac{3}{4}}(s) + \varepsilon^2 t^3 M_{\frac{3}{4}}(s)\Big)}{2\Big(-2M_{-\frac{1}{4}}(s) + \kappa\varepsilon^{\frac{1}{2}} t M_{\frac{1}{4}}(s)\Big)}. \tag{21}$$

(3) *For the solution* $l_\varepsilon(t,\kappa)$, *we have as* $t \to 0$

$$l_\varepsilon(t,\kappa) = ia\varepsilon^{\frac{1}{2}}\left(1 + a\varepsilon^{\frac{1}{2}}t + \left(a\varepsilon^{\frac{1}{2}}t\right)^2 + O\left(\varepsilon^{\frac{1}{2}}t\right)^3\right), \quad a = 2\kappa\Gamma(3/4)/\Gamma(1/4). \tag{22}$$

Proof. Define $w_1(s) = s^{1/8}\Big(aJ_{\frac{1}{4}}(2\sqrt{s}) + bY_{\frac{1}{4}}(2\sqrt{s})\Big)$ for $s \geq 0$. Reference 4, pp. 67–78, shows that the general solution of (7) is given by

$$l_\varepsilon(t) = \frac{-4isw_1'(s)}{tw_1(s)} \tag{23}$$

with arbitrary constants a and b, which are not $\mathbb{R}$-related. If we use $Y_{\frac{1}{4}} = J_{\frac{1}{4}} - \sqrt{2}J_{-\frac{1}{4}}$ and set $\kappa = (a+b)/(\sqrt{2}b) \in (\mathbb{C} \cup \{\infty\})\backslash\mathbb{R}$, the right-hand side of (23) becomes $l_\varepsilon(t,\kappa)$ of (19). Lemma II.2 implies part (1).

To prove part (2), we use the recurrence formula of Bessel functions [see (10.6.5) in DLMF[9]].

Lemma II.4. Let $\mathcal{C}_\nu(z)$ *be any of* $J_\nu(z), Y_\nu(z), H_\nu^{(1)}(z), H_\nu^{(2)}(z)$ *or any nontrivial linear combination of these functions, the coefficients in which are independent of z and ν. Define* $f_\nu(z) = z^p \mathcal{C}_\nu(\lambda z^q)$*, where* $p, q,$ *and* $\lambda \neq 0$ *are real or complex constants; then,*

$$z f_\nu'(z) = \lambda q z^q f_{\nu-1}(z) + (p - \nu q) f_\nu(z) \tag{24}$$

as long as the principal branch is considered for $\mathcal{C}_\nu(z)$*.*

Consider $f_\nu(z)$ for $\mathcal{C}_\nu(z) = aJ_\nu(z) + bY_\nu(z)$ with

$$p = \frac{1}{8}, \quad q = \frac{1}{2}, \quad \nu = \frac{1}{4}, \quad \lambda = 2.$$

Then, $\lambda q = 1, p - \nu q = 0$, and (24) implies that for $w_1(s)$ of (23), we have

$$l_\varepsilon(t) = \frac{-4i}{t} \frac{s\left(s^{\frac{1}{8}} \mathcal{C}_{\frac{1}{4}}(2\sqrt{s})\right)'}{s^{\frac{1}{8}} \mathcal{C}_{\frac{1}{4}}(2\sqrt{s})} = \frac{-4i}{t} \frac{s^{\frac{5}{8}} \mathcal{C}_{-\frac{3}{4}}(2\sqrt{s})}{s^{\frac{1}{8}} \mathcal{C}_{\frac{1}{4}}(2\sqrt{s})}. \tag{25}$$

In the right-hand side of (25), substitute

$$\mathcal{C}_{\frac{1}{4}}(z) = aJ_{\frac{1}{4}}(z) + bY_{\frac{1}{4}}(z) = (a+b)J_{\frac{1}{4}}(z) - \sqrt{2}bJ_{-\frac{1}{4}}(z),$$

$$\mathcal{C}_{-\frac{3}{4}}(z) = aJ_{-\frac{3}{4}}(z) + bY_{-\frac{3}{4}}(z) = (a+b)J_{-\frac{3}{4}}(z) + \sqrt{2}bJ_{\frac{3}{4}}(z)$$

with $z = 2\sqrt{s}$ and reduce by the common factor $\sqrt{2}b$. We have in the denominator

$$\frac{s^{\frac{1}{8}}}{\sqrt{2}b} \mathcal{C}_{\frac{1}{4}}(2\sqrt{s}) = s^{1/8}\left(-J_{-\frac{1}{4}}(2\sqrt{s}) + \kappa J_{\frac{1}{4}}(2\sqrt{s})\right) = w(s, \kappa) \tag{26}$$

and in the numerator

$$\frac{s^{\frac{5}{8}}}{\sqrt{2}b} \mathcal{C}_{-\frac{3}{4}}(2\sqrt{s}) = s^{\frac{5}{8}}\left(\kappa J_{-\frac{3}{4}}(z) + J_{\frac{3}{4}}(z)\right) = \frac{\sqrt{\varepsilon}\kappa t}{2} M_{-\frac{3}{4}}(s) + sM_{\frac{3}{4}}(s). \tag{27}$$

Plugging these in (25), we obtain (20). If we use (17) and the last expression in (27), we obtain (21), which manifests that $l_\varepsilon(t, \kappa)$ is a meromorphic function of t.

To prove part (3), we use (21). The numerator has an asymptotic expansion

$$-i(8\kappa\varepsilon^{\frac{1}{2}}\Gamma(1/4)^{-1} + O(\varepsilon^2 t^3))$$

and the denominator has an asymptotic expansion

$$2(-2\Gamma(3/4)^{-1} + \kappa\varepsilon^{\frac{1}{2}} t\Gamma(5/4)^{-1} + O(\varepsilon^2 t^4))$$

as $t \to 0$; hence,

$$\begin{aligned} l_\varepsilon(t, \kappa) &= \frac{-i8\kappa\varepsilon^{\frac{1}{2}}\Gamma(1/4)^{-1} + O(\varepsilon^2 t^3)}{2(-2\Gamma(3/4)^{-1} + \kappa\varepsilon^{\frac{1}{2}} t\Gamma(5/4)^{-1} + O(\varepsilon^2 t^4))} \\ &= \left(\frac{2i\kappa\varepsilon^{\frac{1}{2}}\Gamma(3/4)}{\Gamma(1/4)} + O(\varepsilon^2 t^3)\right)\left(1 - \frac{2\kappa\varepsilon^{\frac{1}{2}} t\Gamma(3/4)}{\Gamma(1/4)} + O(\varepsilon^2 t^4)\right)^{-1} \\ &= \frac{ia\varepsilon^{\frac{1}{2}}}{1 - a\varepsilon^{\frac{1}{2}} t + O(\varepsilon^2 t^4)} + O(\varepsilon^2 t^3), \quad a = \frac{2\kappa\Gamma(3/4)}{\Gamma(1/4)}. \end{aligned}$$

Statement (3) follows. □

B. The initial condition

Having obtained the general solution $l_\varepsilon(t,\kappa)$ of (7), we need to determine $\kappa = \kappa_\varepsilon$ such that the initial condition $l_\varepsilon(-1/\varepsilon, \kappa_\varepsilon) = 1$ is satisfied. We define

$$l_\varepsilon^*(t) = l_\varepsilon(t,\kappa_\varepsilon) \quad \text{and} \quad \tilde{l}_\varepsilon(t) = -l_\varepsilon^*(-t).$$

We have introduced $\tilde{l}_\varepsilon(t)$ as we want to deal with a positive variable. Then, (21) implies for $t > 0$ that

$$\begin{aligned}\tilde{l}_\varepsilon(t) &= \frac{-i\Big(8(-\kappa_\varepsilon)\varepsilon^{\frac{1}{2}} M_{-\frac{3}{4}}(s) + \varepsilon^2 t^3 M_{\frac{3}{4}}(s)\Big)}{2\Big(-2M_{-\frac{1}{4}}(s) + (-\kappa_\varepsilon)\varepsilon^{\frac{1}{2}} t M_{\frac{1}{4}}(s)\Big)} \\ &= \frac{-4is^{\frac{1}{2}}\Big(-\kappa_\varepsilon J_{-\frac{3}{4}}(2\sqrt{s}) + J_{\frac{3}{4}}(2\sqrt{s})\Big)}{t(-J_{-\frac{1}{4}}(2\sqrt{s}) - \kappa_\varepsilon J_{\frac{1}{4}}(2\sqrt{s}))} \\ &= l_\varepsilon(t,-\kappa_\varepsilon) = -i\frac{\tilde{w}_\varepsilon'(t,-\kappa_\varepsilon)}{\tilde{w}_\varepsilon(t,-\kappa_\varepsilon)}.\end{aligned} \tag{28}$$

Thus, $\tilde{l}_\varepsilon(1/\varepsilon) = -1$ is satisfied if (and only if)

$$-1 = \frac{-4is^{\frac{1}{2}}\Big(-\kappa_\varepsilon J_{-\frac{3}{4}}(2\sqrt{s}) + J_{\frac{3}{4}}(2\sqrt{s})\Big)}{t(-J_{-\frac{1}{4}}(2\sqrt{s}) - \kappa_\varepsilon J_{\frac{1}{4}}(2\sqrt{s}))}\Bigg|_{t=1/\varepsilon} = \frac{i\Big(-\kappa_\varepsilon J_{-\frac{3}{4}}(1/2\varepsilon) + J_{\frac{3}{4}}(1/2\varepsilon)\Big)}{J_{-\frac{1}{4}}(1/2\varepsilon) + \kappa_\varepsilon J_{\frac{1}{4}}(1/2\varepsilon)},$$

where we used $2\sqrt{s} = 1/(2\varepsilon)$ when $t = 1/\varepsilon$ in the last expression. Solving this equation for κ_ε leads to

$$\kappa_\varepsilon = -\frac{J_{-\frac{1}{4}} + iJ_{\frac{3}{4}}}{J_{\frac{1}{4}} - iJ_{-\frac{3}{4}}}\Bigg|_{\frac{1}{2\varepsilon}}. \tag{29}$$

We recall the following special case of (10.17.3) in DLMF.[9]

Lemma II.5. Assume x is real, and let $\omega = x - \frac{1}{2}\nu\pi - \frac{\pi}{4}$. Then, as $x \to \infty$,

$$J_\nu(x) = \sqrt{\frac{2}{\pi x}}\left(\cos\,\omega - \frac{(4\nu^2-1)}{8x}\sin\,\omega + O\left(\frac{1}{x^2}\right)\right). \tag{30}$$

Application of this result to the right-hand side of (29) yields

$$\kappa_\varepsilon = -e^{\frac{i\pi}{4}} + (2\sqrt{2})^{-1}\varepsilon e^{-i\left(\frac{1}{\varepsilon}+\frac{\pi}{4}\right)} + O(\varepsilon^2) = -e^{\frac{i\pi}{4}} + O(\varepsilon), \quad \varepsilon \to 0. \tag{31}$$

We omit the details.

Lemma II.6. The solution $l_\varepsilon^*(t)$ of the initial value problem for the Riccati equation

$$l_\varepsilon^{*'}(t) + il_\varepsilon^*(t)^2 = i\varepsilon^2 t^2, \quad l_\varepsilon^*(-1/\varepsilon) = 1 \tag{32}$$

is given by (20) with κ given by κ_ε of (29),

$$l_\varepsilon^*(t) = \frac{-4is^{\frac{1}{2}}\Big(\kappa_\varepsilon J_{-\frac{3}{4}}(2\sqrt{s}) + J_{\frac{3}{4}}(2\sqrt{s})\Big)}{t(-J_{-\frac{1}{4}}(2\sqrt{s}) + \kappa_\varepsilon J_{\frac{1}{4}}(2\sqrt{s}))}, \quad s = \frac{\varepsilon^2 t^4}{16}, \tag{33}$$

where the principal branch is assumed for Bessel functions.

C. Asymptotic behavior of $l_\varepsilon^*(1/\varepsilon)$ as $\varepsilon \to 0$

Lemma II.7. As $\varepsilon \to 0$, we have

$$l_\varepsilon^*(1/\varepsilon) = \frac{1 - 2\sqrt{2}i\,\cos(1/\varepsilon)}{3 + 2\sqrt{2}\sin(1/\varepsilon)} + O(\varepsilon) \tag{34}$$

and Re $l_\varepsilon^*(1/\varepsilon)$ *oscillates between* $(3+2\sqrt{2})^{-1}$ *and* $(3-2\sqrt{2})^{-1}$ as $\varepsilon\to 0$.

Proof. From (33), we have

$$l_\varepsilon^*(1/\varepsilon) = \frac{-i\Big(\kappa_\varepsilon J_{-\frac{3}{4}}+J_{\frac{3}{4}}\Big)}{-J_{-\frac{1}{4}}+\kappa_\varepsilon J_{\frac{1}{4}}}\Bigg|_{\frac{1}{2\varepsilon}}, \qquad \kappa_\varepsilon = -\frac{J_{-\frac{1}{4}}+iJ_{\frac{3}{4}}}{J_{\frac{1}{4}}-iJ_{-\frac{3}{4}}}\Bigg|_{\frac{1}{2\varepsilon}}. \tag{35}$$

Thus,

$$l_\varepsilon^*(1/\varepsilon) = \frac{2J_{\frac{3}{4}}J_{-\frac{3}{4}}+i\Big(J_{\frac{3}{4}}J_{\frac{1}{4}}-J_{-\frac{1}{4}}J_{-\frac{3}{4}}\Big)}{2J_{-\frac{1}{4}}J_{\frac{1}{4}}+i\Big(J_{\frac{1}{4}}J_{\frac{3}{4}}-J_{-\frac{1}{4}}J_{-\frac{3}{4}}\Big)}\Bigg|_{\frac{1}{2\varepsilon}}, \tag{36}$$

and we may compute the asymptotic value of (36) as $\varepsilon\to 0$ by applying once more (30). This yields (34), and the lemma follows. □

D. The amplitude function $m_\varepsilon(t)$

We next solve initial value problem (9) associated with $l_\varepsilon^*(t)$, which reads

$$\frac{m_\varepsilon'(t)}{m_\varepsilon(t)} = \frac{l_\varepsilon^*(t)}{2i}, \qquad m_\varepsilon(-1/\varepsilon) = \pi^{-1/4}.$$

For the same reason as before, we consider $\tilde m_\varepsilon(t) = m_\varepsilon(-t)$. Expression (28) for $\tilde l_\varepsilon(t)$ implies

$$\frac{\tilde m_\varepsilon'(t)}{\tilde m_\varepsilon(t)} = -\frac{m_\varepsilon'(-t)}{m_\varepsilon(-t)} = -\frac{l_\varepsilon^*(-t)}{2i} = \frac{\tilde l_\varepsilon(t)}{2i} = -\frac{\tilde w_\varepsilon'(t,-\kappa_\varepsilon)}{2\tilde w_\varepsilon(t,-\kappa_\varepsilon)}. \tag{37}$$

Recall (14) and (18) for the definition of $\tilde w_\varepsilon(t,\kappa)$. Integrating (37) yields $\tilde m_\varepsilon(t) = A_\varepsilon \tilde w_\varepsilon(t,-\kappa_\varepsilon)^{-1/2}$ for a constant A_ε for $t>0$, viz.,

$$m_\varepsilon(-t) = A_\varepsilon\Big(-s^{1/8}J_{-\frac{1}{4}}(2\sqrt{s}) - \kappa_\varepsilon s^{1/8}J_{\frac{1}{4}}(2\sqrt{s})\Big)^{-1/2} \tag{38}$$

$$= A_\varepsilon\Big(-M_{-\frac{1}{4}}(s) - \kappa_\varepsilon t M_{\frac{1}{4}}(s)\Big)^{-1/2}. \tag{39}$$

Thus, the initial condition $m_\varepsilon(-1/\varepsilon) = \pi^{-\frac{1}{4}}$ is satisfied if

$$\pi^{-\frac{1}{4}} = A_\varepsilon\Big(-s^{1/8}J_{-\frac{1}{4}}(2\sqrt{s}) - \kappa_\varepsilon s^{1/8}J_{\frac{1}{4}}(2\sqrt{s})\Big)^{-1/2}\Big|_{t=1/\varepsilon}. \tag{40}$$

By virtue of (30) and (31), $(\cdots)$ on the right-hand side is equal to (with $\alpha = \frac{1}{2\varepsilon}-\frac{\pi}{4}$)

$$\frac{2^{\frac{1}{2}}\varepsilon^{\frac{1}{4}}}{\pi^{\frac{1}{2}}}\Big(-\cos\Big(\alpha+\frac{\pi}{8}\Big)+e^{\frac{i\pi}{4}}\cos\Big(\alpha-\frac{\pi}{8}\Big)+O(\varepsilon)\Big) = \frac{\varepsilon^{\frac{1}{4}}}{\pi^{\frac{1}{2}}}e^{-i(\frac{1}{2\varepsilon}-\frac{7\pi}{8})}+O(\varepsilon^{\frac{5}{4}}),$$

and we have

$$A_\varepsilon = \frac{\varepsilon^{\frac{1}{8}}}{\pi^{\frac{1}{2}}}e^{-i(\frac{1}{4\varepsilon}-\frac{7\pi}{16})}(1+O(\varepsilon)). \tag{41}$$

(39) implies that $m_\varepsilon(t)$ is given by changing κ_ε to $-\kappa_\varepsilon$ in the right-hand side of (38) or (39). This proves the first statement of the following lemma.

Lemma II.8.

(1) *The solution of the initial value problem* (9) *associated with* $l_\varepsilon^*(t)$ *is given by*

$$m_\varepsilon(t) = A_\varepsilon\Big(-s^{1/8}J_{-\frac{1}{4}}(2\sqrt{s}) + \kappa_\varepsilon s^{1/8}J_{\frac{1}{4}}(2\sqrt{s})\Big)^{-1/2}, \tag{42}$$

where κ_ε *and* A_ε *are asymptotically given by* (31) *and* (41), *respectively, and the branch of the square root should be chosen such that* $m_\varepsilon(-1/\varepsilon) = \pi^{-\frac{1}{4}}$.

(2) *As* $\varepsilon \to 0$,

$$m_\varepsilon(1/\varepsilon) = \frac{\pi^{-1/4}}{(\sqrt{2}e^{i/\varepsilon} + i)^{1/2}} + O(\varepsilon), \tag{43}$$

where the branch of the square root should be chosen by the continuity.

Proof. By virtue of (40) and (42),

$$m_\varepsilon(1/\varepsilon)^2 = \pi^{-\frac{1}{2}} \frac{-J_{-\frac{1}{4}} - \kappa_\varepsilon J_{\frac{1}{4}}}{-J_{-\frac{1}{4}} + \kappa_\varepsilon J_{\frac{1}{4}}}\Bigg|_{\frac{1}{2\varepsilon}},$$

and we compute the asymptotic value of the right-hand side by using (30). We obtain (43). □

E. Asymptotic behavior at $t = 0$

In Sec. III, we need $l_\varepsilon^*(0)$ and $m_\varepsilon(0)$. We already computed $l_\varepsilon^*(0) = ia\varepsilon^{\frac{1}{2}}$ in (22) where κ in the expression for a should be taken as $\kappa = \kappa_\varepsilon$ [see (31)]. The next lemma immediately follows from (39) or (42).

Lemma II.9. As $t \to 0$, $m_\varepsilon(t)$ *has the following asymptotic expansion, uniformly for* $0 < \varepsilon < 1$:

$$m_\varepsilon(t) = -iA_\varepsilon \Gamma(3/4)^{1/2}\left(1 + \frac{2\Gamma(3/4)}{\Gamma(1/4)}\kappa_\varepsilon t + O(t^2)\right), \tag{44}$$

where A_ε *and* κ_ε *are as in* (41) *and* (31), *respectively.*

Lemma II.9 shows how the adiabatic approximation breaks down as $t \to 0$: The adiabatic approximation would yield $l_\varepsilon(t) = \varepsilon t/2$ for (minus) the exponent of the Gaussian as $\varepsilon \to 0$, whereas the leading term in (22) is $ia\varepsilon^{\frac{1}{2}}$, which does not go to zero as $t \to 0$. The corresponding term of order εt appears only as the second term $ia^2\varepsilon t \sim C^2\varepsilon t/2$, $C = 2\Gamma(3/4)/\Gamma(1/4) \approx 0.676$. The state at time $t = 0$, $v_\varepsilon(0, x)$, is a Gaussian, which is a result of general theorems (see Ref. 7), but the speed of spreading is $C\varepsilon^{1/4}\sqrt{t}$ times slower than the one given by the adiabatic approximation, and at time zero, it remains as a finite Gaussian of size $C\varepsilon^{-1/4}$, whereas the adiabatic approximation gives a completely flat Gaussian.

F. Behavior of $v_\varepsilon(1/\varepsilon)$ as $\varepsilon \to 0$ and the survival probability

The following theorem states the main result of this section for the case $L = 0$. The theorem explicitly exhibits that the state at the microscopic time $1/\varepsilon$, when the Hamiltonian returns to the initial $-(1/2)d^2/dx^2 + (1/2)x^2$, is highly oscillating as $\varepsilon \to 0$ and the adiabatic approximation is completely broken down.

We introduce the notation for the leading terms in asymptotic expansions (34) and (43). We define

$$l_{\varepsilon,0}^*(1/\varepsilon) = \frac{1 - 2\sqrt{2}i\cos(1/\varepsilon)}{3 + 2\sqrt{2}\sin(1/\varepsilon)}, \tag{45}$$

$$m_{\varepsilon,0}(1/\varepsilon) = \frac{\pi^{-1/4}}{(\sqrt{2}e^{i/\varepsilon} + i)^{1/2}}. \tag{46}$$

We have proven the following theorem:

Theorem II.10.

(1) *Let* $l_{\varepsilon,0}^*(1/\varepsilon)$ *and* $m_{\varepsilon,0}(1/\varepsilon)$ *be given by* (45) *and* (46), *respectively. Then, the solution* $v_\varepsilon(t, x)$ *of the initial value problem* (6) *satisfies, as* $\varepsilon \to 0$,

$$\|v_\varepsilon(1/\varepsilon, x) - m_{\varepsilon,0}(1/\varepsilon)e^{-l_{\varepsilon,0}^*(1/\varepsilon)x^2/2}\| \le C\varepsilon. \tag{47}$$

We have

$$|m_{\varepsilon,0}(1/\varepsilon)|^4 = \pi^{-1}\left(3 + 2\sqrt{2}\sin(1/\varepsilon)\right)^{-1} = \pi^{-1}\mathrm{Re}\ l_{\varepsilon,0}^*(1/\varepsilon). \tag{48}$$

(2) *The survival probability of the ground state* $\varphi_0(x) = \pi^{-\frac{1}{4}}e^{-x^2/2}$ *at time* $1/\varepsilon$ *is equal to* $1/\sqrt{2} + O(\varepsilon)$.

Remark II.11. The survival probability in part (2) was also computed in Theorem 1 of Bachmann *et al.*[1]

Proof. Since $\mathrm{Re}\ \ell_\varepsilon^*(1/\varepsilon) \ge (3 + 2\sqrt{2})^{-1}$, part (1) is obvious. We only prove (2). Using (47) and explicitly computing the Gaussian integral, we obtain

$$\langle v_\varepsilon(1/\varepsilon,x),\varphi_0(x)\rangle = \int_{\mathbb{R}} \pi^{-\frac{1}{4}} e^{-x^2/2} m_{\varepsilon,0}(1/\varepsilon) e^{-l^*_{\varepsilon,0}(1/\varepsilon)x^2/2} dx + O(\varepsilon)$$
$$= \frac{\sqrt{2}\pi^{1/4} m_{\varepsilon,0}(1/\varepsilon)}{(1+l^*_{\varepsilon,0}(1/\varepsilon))^{1/2}} + O(\varepsilon).$$

Insert the expressions from (45) and (46). Since

$$(\sqrt{2}e^{i/\varepsilon}+i)\left(1+\frac{1-2\sqrt{2}i\cos(1/\varepsilon)}{3+2\sqrt{2}\sin(1/\varepsilon)}\right) = e^{i/\varepsilon}\frac{6\sqrt{2}+8\sin(1/\varepsilon)}{3+2\sqrt{2}\sin(1/\varepsilon)} = e^{i/\varepsilon}2\sqrt{2},$$

we conclude that

$$|\langle v_\varepsilon(1/\varepsilon,x),\varphi_0(x)\rangle|^2 = \frac{2(3+2\sqrt{2}\sin(1/\varepsilon))}{6\sqrt{2}+8\sin(1/\varepsilon)} + O(\varepsilon) = \frac{1}{\sqrt{2}} + O(\varepsilon).$$

□

III. THE CASE $L > 0$

We next study the case $L > 0$ and examine how the asymptotic behavior as $\varepsilon \to 0$ of the solution depends on the macroscopic length L of time that the particle has spent in the continuum of $-(1/2)\partial_x^2$. We let $v_\varepsilon(t,x)$ be the solution of the initial value problem (6) with $L > 0$. Then, by translating in time the result for the case $L = 0$ by $-L/\varepsilon$, we see from (22) with $\kappa = \kappa_\varepsilon$ and (44) that

$$v_\varepsilon(-L/\varepsilon,x) = -iA_\varepsilon\,\Gamma(3/4)^{1/2} e^{-ia_\varepsilon \varepsilon^{1/2}x^2/2}, \quad a_\varepsilon = \frac{2\Gamma(3/4)}{\Gamma(1/4)}\kappa_\varepsilon.$$

A. Solution at the time exiting the continuum

We may explicitly compute

$$v_\varepsilon(L/\varepsilon,x) = \left(e^{-2iLH_0/\varepsilon} v_\varepsilon(-L/\varepsilon)\right)(x)$$
$$= \frac{-iA_\varepsilon\,\Gamma(3/4)^{1/2} e^{-\frac{\pi i}{4}}}{(4\pi L/\varepsilon)^{1/2}} \int_{\mathbb{R}} e^{\frac{i\varepsilon(x-y)^2}{4L} - i\frac{a_\varepsilon \varepsilon^{1/2} y^2}{2}} dy$$
$$= \frac{-A_\varepsilon \Gamma(3/4)^{\frac{1}{2}} \varepsilon^{\frac{1}{4}}}{(-2a_\varepsilon L+\sqrt{\varepsilon})^{1/2}} e^{\frac{-i\varepsilon a_\varepsilon}{2(-2a_\varepsilon L+\sqrt{\varepsilon})}x^2}. \tag{49}$$

B. Solution after the particle exits the continuum

We want to evaluate at time $t = \frac{L+1}{\varepsilon}$ the solution of

$$i\partial_t v_\varepsilon(t,x) = -\frac{1}{2}\partial_x^2 v_\varepsilon + \frac{(t-L/\varepsilon)^2\varepsilon^2 x^2}{2} v_\varepsilon$$

when $v_\varepsilon(L/\varepsilon,x)$ is given by (49). Translation of t by L/ε once again shows that $v_\varepsilon((L+1)/\varepsilon,x) = z_\varepsilon(1/\varepsilon,x)$, where $z_\varepsilon(t,x)$ is the solution of

$$i\partial_t z_\varepsilon(t,x) = -\frac{1}{2}\partial_x^2 z_\varepsilon + \frac{t^2\varepsilon^2x^2}{2} z_\varepsilon, \tag{50}$$

$$z_\varepsilon(0,x) = \frac{-A_\varepsilon \Gamma(3/4)^{\frac{1}{2}}\varepsilon^{\frac{1}{4}}}{(-2a_\varepsilon L+\sqrt{\varepsilon})^{1/2}} e^{\frac{-i\varepsilon a_\varepsilon}{2(-2a_\varepsilon L+\sqrt{\varepsilon})}x^2}. \tag{51}$$

We know from (20) that $z_\varepsilon(t,x)$ is of the form

$$z_\varepsilon(t,x) = m^*_\varepsilon(t) e^{-l_\varepsilon(t,\gamma)x^2/2},$$

where $l_\varepsilon(t,\gamma)$ and $m^*_\varepsilon(t)$ are given by (20) and (42), respectively, with γ in place of κ and B_ε in place of A_ε, in particular,

$$l_\varepsilon(t,\gamma) = \frac{-4is^{\frac{1}{2}}\left(\gamma J_{-\frac{3}{4}}(2\sqrt{s}) + J_{\frac{3}{4}}(2\sqrt{s})\right)}{t\left(-J_{-\frac{1}{4}}(2\sqrt{s}) + \gamma J_{\frac{1}{4}}(2\sqrt{s})\right)}, \tag{52}$$

$$m_\varepsilon^*(t) = B_\varepsilon\Big(-s^{1/8}J_{-\frac{1}{4}}(2\sqrt{s}) + \gamma s^{1/8}J_{\frac{1}{4}}(2\sqrt{s})\Big)^{-1/2}. \tag{53}$$

We will choose γ and B_ε such that initial condition (51) is met, viz.,

$$l_\varepsilon(0,\gamma) = \frac{i\varepsilon a_\varepsilon}{-2a_\varepsilon L + \sqrt{\varepsilon}}, \tag{54}$$

$$m_\varepsilon^*(0) = \frac{-A_\varepsilon\Gamma(3/4)^{\frac{1}{2}}\varepsilon^{\frac{1}{4}}}{(-2a_\varepsilon L + \sqrt{\varepsilon})^{1/2}}. \tag{55}$$

By virtue of (16), we may evaluate $l_\varepsilon(0,\gamma)$ of (52) and $m_\varepsilon^*(0)$ of (53),

$$l_\varepsilon(0,\gamma) = \frac{2i\gamma\varepsilon^{\frac{1}{2}}\Gamma(3/4)}{\Gamma(1/4)}, \quad m_\varepsilon^*(0) = -iB_\varepsilon\Gamma(3/4)^{1/2}. \tag{56}$$

Equating the right-hand sides of (54) and (55) with those of (56), we have

$$\gamma = \frac{\varepsilon^{\frac{1}{2}}\kappa_\varepsilon}{-2a_\varepsilon L + \sqrt{\varepsilon}}, \quad B_\varepsilon = \frac{-iA_\varepsilon\varepsilon^{\frac{1}{4}}}{(-2a_\varepsilon L + \sqrt{\varepsilon})^{1/2}}. \tag{57}$$

Hereafter, we write $C_1 = \Gamma(3/4)/\Gamma(1/4)$ so that $a_\varepsilon = 2\kappa_\varepsilon C_1$. Note that $\gamma = \kappa_\varepsilon$ and $B_\varepsilon = A_\varepsilon$ when $L = 0$ as they should be.

C. Solution when the Hamiltonian returns to $-(1/2)d^2/dx^2 + x^2/2$

We study the behavior as $\varepsilon \to 0$ of $l_\varepsilon(1/\varepsilon,\gamma)$ and $m_\varepsilon(1/\varepsilon)$. They are given by

$$l_\varepsilon(1/\varepsilon,\gamma) = \frac{-i\Big(\gamma J_{-\frac{3}{4}}(z) + J_{\frac{3}{4}}(z)\Big)}{\Big(-J_{-\frac{1}{4}}(z) + \gamma J_{\frac{1}{4}}(z)\Big)}\Bigg|_{z=\frac{1}{2\varepsilon}}, \tag{58}$$

$$m_\varepsilon(1/\varepsilon) = B_\varepsilon\Big(-(z/2)^{1/4}J_{-\frac{1}{4}}(z) + \gamma(z/2)^{1/4}J_{\frac{1}{4}}(z)\Big)^{-1/2}\Big|_{z=\frac{1}{2\varepsilon}}. \tag{59}$$

We substitute the first of (57) for γ, which yields

$$l_\varepsilon(1/\varepsilon,\gamma) = i\frac{-4\kappa_\varepsilon C_1 L J_{\frac{3}{4}}(z) + \varepsilon^{1/2}\Big(\kappa_\varepsilon J_{-\frac{3}{4}}(z) + J_{\frac{3}{4}}(z)\Big)}{-4\kappa_\varepsilon C_1 L J_{-\frac{1}{4}}(z) - \varepsilon^{1/2}\Big(\kappa_\varepsilon J_{\frac{1}{4}}(z) - J_{-\frac{1}{4}}(z)\Big)}\Bigg|_{z=\frac{1}{2\varepsilon}}. \tag{60}$$

We substitute $\kappa_\varepsilon = -e^{\frac{i\pi}{4}} + O(\varepsilon)$ in (60) and use (30). Denote

$$L_1 = -4C_1 L, \quad \alpha = (2\varepsilon)^{-1} - 4^{-1}\pi.$$

Then, as $\varepsilon \to 0$, $l_\varepsilon(1/\varepsilon)$ (omitting γ in the notation) is asymptotically equal to

$$i\,\frac{-L_1 e^{\frac{i\pi}{4}}\cos(\alpha - \frac{3\pi}{8}) + \varepsilon^{1/2}\Big(-e^{\frac{i\pi}{4}}\cos(\alpha + \frac{3\pi}{8}) + \cos(\alpha - \frac{3\pi}{8})\Big) + O(\varepsilon)}{-L_1 e^{\frac{i\pi}{4}}\cos(\alpha + \frac{\pi}{8}) + \varepsilon^{1/2}\Big(\cos(\alpha + \frac{\pi}{8}) + e^{\frac{i\pi}{4}}\cos(\alpha - \frac{\pi}{8})\Big) + O(\varepsilon)}$$

$$= i\,\frac{-L_1\sin(\frac{1}{2\varepsilon} - \frac{\pi}{8}) + \varepsilon^{1/2}\Big(e^{\frac{-i\pi}{4}}\sin(\frac{1}{2\varepsilon} - \frac{\pi}{8}) + \sin(\frac{1}{2\varepsilon} - \frac{3\pi}{8})\Big) + O(\varepsilon)}{-L_1\cos(\frac{1}{2\varepsilon} - \frac{\pi}{8}) + \varepsilon^{1/2}\Big(e^{\frac{-i\pi}{4}}\cos(\frac{1}{2\varepsilon} - \frac{\pi}{8}) + \cos(\frac{1}{2\varepsilon} - \frac{3\pi}{8})\Big) + O(\varepsilon)}.$$

After a simple but tedious computation, we simplify the equation above and obtain the following lemma.

Lemma III.1. Define $\rho = \frac{1}{2\varepsilon} - \frac{\pi}{8}$ and $B = L_1 - \sqrt{2\varepsilon}$. As $\varepsilon \to 0$, we have

$$l_\varepsilon(1/\varepsilon) = \frac{\varepsilon + i\Big(B^2\sin 2\rho + \sqrt{2\varepsilon}B\cos 2\rho\Big) + O(\varepsilon)}{B^2 + \varepsilon + B^2\cos 2\rho - \sqrt{2\varepsilon}B\sin 2\rho + O(\varepsilon)}. \tag{61}$$

We note that in the denominator, we have

$$A(\varepsilon) \equiv B^2 + \varepsilon + B^2 \cos 2\rho - \sqrt{2\varepsilon}B \sin 2\rho$$
$$= B^2\left(1 + \frac{\varepsilon}{B^2} + \left(1 + \frac{2\varepsilon}{B^2}\right)^{\frac{1}{2}} \cos(2\rho + \beta)\right) \geq C\varepsilon^2, \tag{62}$$

where

$$\sin \beta = \frac{\sqrt{2\varepsilon}}{(B^2 + 2\varepsilon)^{\frac{1}{2}}}.$$

However, $A(\varepsilon) + O(\varepsilon)$ can be controlled only when $A(\varepsilon) \geq C\varepsilon^{1-\delta}$ for $\delta > 0$, and this requires

$$\cos(2\rho + \beta) > -1 + C\varepsilon^{1-\delta}, \quad \delta > 0, \tag{63}$$

in which case we have indeed

$$\frac{A(\varepsilon)}{B^2} = 1 + \frac{\varepsilon}{B^2} + \left(1 + \frac{2\varepsilon}{B^2}\right)^{\frac{1}{2}} \cos(2\rho + \beta)$$
$$> 1 + \frac{\varepsilon}{B^2} + \left(1 + \frac{\varepsilon}{B^2} - O(\varepsilon^2)\right)(-1 + C\varepsilon^{1-\delta}) \geq C\varepsilon^{1-\delta}. \tag{64}$$

Let $\Omega \subset (0, 1)$ be the set of ε that does not satisfy (63). Then, Taylor's formula implies for some $C > 0$ that

$$\Omega \subset \bigcup_{n=0}^{\infty} \left\{\varepsilon > 0 : \left|\frac{1}{\varepsilon} - \frac{\pi}{4} + \beta - (2n+1)\pi\right| < C\varepsilon^{(1-\delta)/2}\right\}. \tag{65}$$

The definition of β and Taylor's formula imply

$$\sin \beta = \beta - O(\beta^3) = \frac{\sqrt{2\varepsilon}}{L_1}\left(1 + \frac{2\sqrt{2\varepsilon}}{L_1} + \frac{4\varepsilon}{L_1^2}\right)^{-\frac{1}{2}} = \frac{\sqrt{2\varepsilon}}{L_1}\left(1 - \frac{\sqrt{2\varepsilon}}{L_1} + O(\varepsilon)\right),$$

and as $\varepsilon \to 0$,

$$\beta = \frac{\sqrt{2\varepsilon}}{L_1} + O(\varepsilon). \tag{66}$$

(66) implies that $\varepsilon > 0$ that satisfies (65) must satisfy

$$\left|\frac{1}{\varepsilon} - \frac{\pi}{4} - \frac{\sqrt{2\varepsilon}}{L_1} - (2n+1)\pi\right| < C\varepsilon^{(1-\delta)/2} \tag{67}$$

for some n and (another) constant $C > 0$. We want to solve (67) for ε in terms of n. (67) is equivalent to

$$\left|\varepsilon - \frac{1}{\frac{\pi}{4} - \frac{2\sqrt{\varepsilon}}{L_1} + (2n+1)\pi}\right| < \frac{C\varepsilon^{\frac{3-\delta}{2}}}{\frac{\pi}{4} - \frac{2\sqrt{\varepsilon}}{L_1} + (2n+1)\pi}. \tag{68}$$

For small $\varepsilon > 0$ or for large n, this implies $(4n\pi)^{-1} \leq |\varepsilon| \leq C(n\pi)^{-1}$ and

$$\left|\varepsilon - \frac{1}{\frac{\pi}{4} - \frac{2\sqrt{\varepsilon}}{L_1} + (2n+1)\pi}\right| < Cn^{-(5-\delta)/2}.$$

Define

$$\varepsilon(n) = \left(\frac{\pi}{4} + (2n+1)\pi\right)^{-\frac{1}{2}}. \tag{69}$$

Then,

$$|\sqrt{\varepsilon} - \varepsilon(n)| = \frac{|\varepsilon - \varepsilon(n)^2|}{\sqrt{\varepsilon} + \varepsilon(n)} \le Cn^{-\frac{3-\delta}{2}}$$

and

$$\left| \frac{1}{\frac{\pi}{4} - \frac{2\sqrt{\varepsilon}}{L_1} + (2n+1)\pi} - \frac{1}{\frac{\pi}{4} - \frac{2\varepsilon(n)}{L_1} + (2n+1)\pi} \right| \le Cn^{-\frac{7-\delta}{2}}.$$

In this way, we have shown that for some $C > 0$,

$$\Omega \subset \tilde{\Omega} = \bigcup_{n=0}^{\infty} \left\{ \varepsilon : \left| \varepsilon - \frac{1}{\frac{\pi}{4} - \frac{2\varepsilon(n)}{L_1} + (2n+1)\pi} \right| < Cn^{-(5-\delta)/2} \right\}. \tag{70}$$

Thus, we have obtained the following theorem.

Theorem III.2. *Let* $0 < \delta < 1$, $B = L_1 - \sqrt{2\varepsilon}$, $\varepsilon(n)$ *be defined by* (69), *and* $\tilde{\Omega}$ *be defined by* (70) *with a suitable constant* $C > 0$. *Denote* $\rho = \frac{1}{2\varepsilon} - \frac{\pi}{8}$. *Then, for* $\varepsilon \notin \tilde{\Omega}$, $B^2 + \varepsilon + B^2 \cos 2\rho - \sqrt{2\varepsilon}B \sin 2\rho \ge C\varepsilon^{1-\delta}$, *and as* $\varepsilon \to 0$,

$$l_\varepsilon(1/\varepsilon) = \frac{\varepsilon + i\big(B^2 \sin 2\rho + \sqrt{2\varepsilon}B \cos 2\rho\big) + O(\varepsilon)}{B^2 + \varepsilon + B^2 \cos 2\rho - \sqrt{2\varepsilon}B \sin 2\rho}\Big(1 + O(\varepsilon^\delta)\Big). \tag{71}$$

We note that $\mathrm{Re}\, l_\varepsilon(1/\varepsilon) \le C\varepsilon^\delta$ and $|v_\varepsilon(1/\varepsilon, x)| \le C_\varepsilon \exp(-C\varepsilon^\delta x^2/L)$ for $\varepsilon \notin \tilde{\Omega}$. Recall that the free Schrödinger operator $-\partial_x^2$ has a zero resonance with resonant function 1. It follows that, as $\varepsilon \notin \tilde{\Omega}$ approaches 0, $v_\varepsilon(1/\varepsilon, x)$ approaches an oscillating function of the magnitude of the resonant function of $-\partial_x^2$ on every compact interval of $\mathbb{R}$, and it does so faster when the length L becomes longer, $2L$ being the time the particle stays as a free particle. Here, the behavior as $\varepsilon \to 0$ of the imaginary part of $l_\varepsilon(1/\varepsilon)$ heavily depends on how ε approaches 0; however, we shall not pursue this point any further here.

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Хугацаанаас хамаарсан потенциалтай Шредингерийн тэгшитгэлийн аналитик шийдийн талаар

Э. Ганхөлөг

Удиртгал

Хугацаанаас хамаарсан потенциал бүхий нэг хэмжээст Шредингерийн тэгшитгэл авч үзье.

$$i\varepsilon\partial_t u_\varepsilon = -\frac{1}{2}\partial_x^2 u_\varepsilon + V(t,x)u_\varepsilon, \quad u_\varepsilon(-L-1,x) = \pi^{-\frac{1}{4}}e^{-\frac{x^2}{2}}$$

энд $L > 0,\ \ 0 < \varepsilon \ll 1$ ба

$$V(t,x) = \begin{cases} (t+L)^2x^2/2, & t < -L, \\ 0, & -L \le t \le L, \\ (t-L)^2x^2/2, & t > L. \end{cases}$$

Энэ тэгшитгэл нь аналитик шийдтэй бөгөөд уг шийд нь $u_\varepsilon(t,x) = m(t)e^{-l(t)\frac{x^2}{2}}$ хэлбэртэй. $m(t)$ болон $l(t)$ функцүүд нь I төрлийн Бесселийн функцээр илэрхийлэгдэнэ. **A. Galtbayar, A. Jensen, K. Yajima** нарын ***"A solvable model of the breakdown of the adiabatic approximation"*** ажилд харуулсан байдаг. Би уг ажлыг өргөтгөж, шийдийг ε параметрийн хувьд аналитик задаргааг шууд нарийвчлалтай тооцоолж $\varepsilon \to 0$ үеийн аналитик чанарыг судлахад:

- $l(t)$ функцийн бодит хэсэг $(0,1]$ хооронд $|\sin\frac{1}{\varepsilon}|$ төрлийн хэлбэлзэл хийж байсан бол хуурмаг хэсэг $(-\infty,\infty)$ хооронд $\cot\frac{1}{\varepsilon}$ төрлийн хэлбэлзэл хийнэ.
- Хугацаанаас хамаарсан потенциалтай Шредингерийн тэгшитгэлийн шийд анхны төлөв байдлаа хадгалах магадлал 0-рүү тэмүүлнэ.

Түлхүүр үг: `Хугацаанаас хамаарсан потенциал, Адиабат теорем`

1 Оршил

Хугацаанаас хамаарсан гармоник потенциал бүхий $i\varepsilon\partial_t u(t) = H(t)u(t)$ Шредингерийн тэгшитгэлийг өгөгдсөн $L > 0, \ \ 0 < \varepsilon \ll 1$ хувьд авч үзье.

$$i\varepsilon\partial_t u_\varepsilon = -\frac{1}{2}\partial_x^2 u_\varepsilon + V(t,x)u_\varepsilon, \quad u_\varepsilon(-L-1,x) = \varphi_0(x) \tag{1.1}$$

энд

$$V(t,x) = \begin{cases} (t+L)^2x^2/2, & t < -L, \\ 0, & -L \le t \le L, \\ (t-L)^2x^2/2, & t > L \end{cases}$$

ба Гамильтон $H(-L-1) = -\frac{1}{2}\Delta + \frac{x^2}{2}$-ийн энергийн хамгийн бага түвшин дэх хувийн функцээр (I.1) тэгшитгэлийн анхны нөхцөлийг $\varphi_0(x) = \pi^{-\frac{1}{4}}e^{-\frac{x^2}{2}}$ байхаар өгнө.

1.1 Шредингерийн тэгшитгэлийн шийд

(I.1) тэгшитгэлийн шийдийг судлах нь $t \to t\varepsilon$ орлуулгаар үүсэх эквивалент тэгшитгэлийг судалсантай адилхан.

$$i\partial_t v_\varepsilon = -(1/2)\partial_x^2 v_\varepsilon + V(\varepsilon t,x)v_\varepsilon, \quad v_\varepsilon(-(L+1)/\varepsilon) = \pi^{-\frac{1}{4}}e^{-\frac{x^2}{2}}. \tag{1.2}$$

энд

$$V(\varepsilon t,x) = \begin{cases} (\varepsilon t+L)^2x^2/2, & t < -\frac{L}{\varepsilon}, \\ 0, & -\frac{L}{\varepsilon} \le t \le \frac{L}{\varepsilon}, \\ (\varepsilon t-L)^2x^2/2, & t > \frac{L}{\varepsilon}. \end{cases}$$

(I.2) тэгшитгэл нь хугацаа t аль ч агшинд байсан $v(t,x) = m_\varepsilon(t)e^{-l_\varepsilon(t)\frac{x^2}{2}}$ хэлбэрийн аналитик шийдтэй.

Дээрх Шредингерийн тэгшитгэлийг хугацааны хувьд $-(L+1)/\varepsilon < t < -L/\varepsilon$ завсарт сонирхъё.

Тэгвэл $l_\varepsilon(t)$-ийн хувьд дараах $l_\varepsilon(-(L+1)/\varepsilon) = 1$ анхны нөхцөлтэй Риккатийн тэгшитгэл үүснэ

$$l_\varepsilon'(t) + il_\varepsilon(t)^2 = i(\varepsilon t+L)^2. \tag{1.3}$$

Харин $l_\varepsilon(t)$, $m_\varepsilon(t)$ хувьд $m_\varepsilon(-(L+1)/\varepsilon) = 1/\pi^{\frac{1}{4}}$ анхны нөхцөл бүхий доорх тэгшитгэл үүснэ

$$im'_\varepsilon(t) = \frac{1}{2} m_\varepsilon(t) l_\varepsilon(t). \tag{1.4}$$

(I.3) болон (I.4) тэгшитгэлийг $\tau = t + L/\varepsilon$ орлуулгаар хугацааны хувьд шилжүүлэлт хийвэл: $\tau \in (-1/\varepsilon, 0)$

$$l'_\varepsilon(\tau) + il_\varepsilon(\tau)^2 = i(\varepsilon\tau)^2, \quad l_\varepsilon(-1/\varepsilon) = 1 \tag{1.5}$$

болон

$$im_\varepsilon(\tau) = \frac{1}{2} m_\varepsilon(\tau) l_\varepsilon(\tau), \quad m_\varepsilon(-1/\varepsilon) = 1/\pi^{\frac{1}{4}} \tag{1.6}$$

тэгшитгэлүүд үүсэж гарна. Энэ орлуулга нь $L = 0$ үед үүсэх тэгшитгэлтэй эквивалент байна. Иймд $L = 0$ шийдээс $L > 0$ шийдийг байгуулна. Цаашилбал "Адиабат теорем" биелэх эсэхийг судална.

1.2 Судлагдсан байдал

1916 онд Эренфест хугацаанаас хамаарсан ”Адиабат теорем” санааг дэвшүүлсэн. Энэ санаан дээр үндэслэн 1928 онд М. Борн болон В.А. Фок [2] нар квант механик дахь давхардсан хувийн утгагүй Гамильтон системийг судалсан анхдагч ажил байв. Энэхүү ажлаар ”Адиабат теорем”-ийн тухай ерөнхий ойлголтыг тайлбарласан байдаг. Уг ажлыг 1950 онд Т. Като [3] ажлаараа өргөтгөн давхардсан хувийн утгатай системийг судалсан байдаг. ”Адиабат теорем”-ийн гол санаа нь квант системийн анхны төлөвийг Гамильтон операторын хамгийн бага хувийн утгад харгалзах хувийн функцээр (Хувийн функцүүдийн огторгуй дээрх функц) өгөөд системийн төлөв байдлыг хугацааны хувьд Гамильтон оператор хэрхэн өөрчлөгдөж байгааг судалдаг. Хэрэв квант системд ”Адиабат теорем” биелдэг бол уг системийн төлөв байдлын өөрчлөлтийг Гамильтон операторын хувийн утга, хувийн функцээр дамжуулан мэдэх боломжтой.

Гэвч хугацаанаас хамаарсан Гамильтон оператортай Шредингерийн тэгшитгэлийн шийдийг олох нь багагүй хүндрэлтэй.

2 $L = 0$ үеийн шийд

Тэмдэглэгээг эвтэйхэн байлгах үүднээс (I.5) тэгшитгэлийн $\tau = t$ гэсэн орлуулга хийе.

$$l_\varepsilon^{'}(t) + il_\varepsilon(t)^2 = i\varepsilon^2 t^2. \tag{2.1}$$

Тэгвэл Риккатийн тэгшитгэлийн ерөнхий шийд (II.2) хэлбэртэй

$$l_\varepsilon(t) = -i\frac{u_\varepsilon^{'}(t)}{u_\varepsilon(t)} \tag{2.2}$$

$\tau = t\sqrt{\varepsilon}$, a, b тогтмол тоонуудын хувьд

$$u_\varepsilon(t) = a\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(k-\frac{1}{4}+1)}\left(\frac{\tau}{2}\right)^{4k} + b\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(k+\frac{1}{4}+1)}\left(\frac{\tau}{2}\right)^{4k+1}$$

байна. Одоо (II.3) байдлаар тодорхойлогдох Бессел функцийн цуваа хэлбэрийн тодорхойлолтыг ашиглана.

$$J_\nu(z) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!\,\Gamma(k+\nu+1)}\left(\frac{z}{2}\right)^{2k+\nu} \tag{2.3}$$

$u_\varepsilon(t)$ функцийг Бессел функцээр илэрхийлбэл:

$$\begin{aligned} u_\varepsilon(t) &= a\left(\frac{\tau^2}{4}\right)^{\frac{1}{4}}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(k-\frac{1}{4}+1)}\left(\frac{\tau^2}{4}\right)^{2k-\frac{1}{4}} \\ &+ b\left(\frac{\tau^2}{4}\right)^{-\frac{1}{4}}\frac{\tau}{2}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(k+\frac{1}{4}+1)}\left(\frac{\tau^2}{4}\right)^{2k+\frac{1}{4}} \\ &= a\left(\frac{\tau}{2}\right)^{\frac{1}{2}}J_{-\frac{1}{4}}\left(\frac{\tau^2}{2}\right) + b\left(\frac{\tau}{2}\right)^{\frac{1}{2}}J_{\frac{1}{4}}\left(\frac{\tau^2}{2}\right) \end{aligned} \tag{2.4}$$

болно.

Одоо $u_\varepsilon(t)$ функцийн уламжлалыг тооцоход:

$$\begin{aligned} u'_\varepsilon(t) &= a\sqrt{\varepsilon}\sum_{k=1}^{\infty}\frac{4k(-1)^k}{2k!\Gamma(k-\frac{1}{4}+1)}\left(\frac{\tau}{2}\right)^{4k-1} \\ &+ b\sqrt{\varepsilon}\sum_{k=0}^{\infty}\frac{(4k+1)(-1)^k}{2k!\Gamma(k+\frac{1}{4}+1)}\left(\frac{\tau}{2}\right)^{4k} \\ &= 2\sqrt{\varepsilon}\left(\frac{\tau}{2}\right)^{\frac{3}{2}}\left[-aJ_{\frac{3}{4}}\left(\frac{\tau^2}{2}\right)+bJ_{-\frac{3}{4}}\left(\frac{\tau^2}{2}\right)\right] \text{ гэж олдсон.} \end{aligned}$$

Эндээс $l_\varepsilon(t)$ функцийг Бессел функцээр илэрхийлбэл:

$$\begin{aligned} l_\varepsilon(t) &= -i\frac{u'_\varepsilon(t)}{u_\varepsilon(t)} = \\ &= \frac{-i\sqrt{\varepsilon}\tau\left[-aJ_{\frac{3}{4}}\left(\frac{\tau^2}{2}\right)+bJ_{-\frac{3}{4}}\left(\frac{\tau^2}{2}\right)\right]}{aJ_{-\frac{1}{4}}\left(\frac{\tau^2}{2}\right)+bJ_{\frac{1}{4}}\left(\frac{\tau^2}{2}\right)} \text{ болно.} \end{aligned}$$

Одоо $k_\varepsilon = -b/a$ гэсэн тэмдэглэгээ хийж Риккатийн тэгшитгэлийн ерөнхий шийдийг бичвэл:

$$l_\varepsilon(t) = \frac{-i\sqrt{\varepsilon}\tau\left[J_{\frac{3}{4}}\left(\frac{\tau^2}{2}\right)+k_\varepsilon J_{-\frac{3}{4}}\left(\frac{\tau^2}{2}\right)\right]}{-J_{-\frac{1}{4}}\left(\frac{\tau^2}{2}\right)+k_\varepsilon J_{\frac{1}{4}}\left(\frac{\tau^2}{2}\right)} \text{ байна.}$$

(II.1) тэгшитгэлийн анхны нөхцөл t-ийн сөрөн утганд байгаа. Иймд ойлгомжтой байлгах үүднээс $t > 0$ гэж үзээд Риккатын тэгшитгэлийн ерөнхий шийдийн хугацааны сөрөг утганд буюу $l_\varepsilon(-t)$ байх үед :

$$l_\varepsilon(-t) = \frac{-i\sqrt{\varepsilon}\tau\left[J_{\frac{3}{4}}\left(\frac{\tau^2}{2}\right)-k_\varepsilon J_{-\frac{3}{4}}\left(\frac{\tau^2}{2}\right)\right]}{J_{-\frac{1}{4}}\left(\frac{\tau^2}{2}\right)+k_\varepsilon J_{\frac{1}{4}}\left(\frac{\tau^2}{2}\right)} \text{ болно.}$$

$1 = \frac{-i\left[J_{\frac{3}{4}}\left(\frac{1}{2\varepsilon}\right)-k_\varepsilon J_{-\frac{3}{4}}\left(\frac{1}{2\varepsilon}\right)\right]}{J_{-\frac{1}{4}}\left(\frac{1}{2\varepsilon}\right)+k_\varepsilon J_{\frac{1}{4}}\left(\frac{1}{2\varepsilon}\right)}$ болно. Эндээс k_ε тогтмолыг

$$\begin{aligned} k_\varepsilon &= -\left.\frac{J_{-\frac{1}{4}}+iJ_{\frac{3}{4}}}{J_{\frac{1}{4}}-iJ_{-\frac{3}{4}}}\right|_{\frac{1}{2\varepsilon}} \\ &= -\left.\frac{J_{-\frac{1}{4}}J_{\frac{1}{4}}-J_{-\frac{3}{4}}J_{\frac{3}{4}}+i(J_{-\frac{1}{4}}J_{-\frac{3}{4}}+J_{\frac{3}{4}}J_{\frac{1}{4}})}{J^2_{\frac{1}{4}}+J^2_{-\frac{3}{4}}}\right|_{\frac{1}{2\varepsilon}} \end{aligned}$$

томьёогоор олно.

Хангалттай том x-ийн хувьд $\omega = x - \frac{1}{2}\nu\pi - \frac{\pi}{4}$ гэсэн тэмдэглэгээ хийвэл Бессел функц

$$J_\nu(x) = \sqrt{\frac{2}{\pi x}}\Big(\cos\,\omega - \frac{(4\nu^2-1)}{8x}\sin\,\omega + O\big(1/x^2\big)\Big) \tag{2.5}$$

байдаг([4, 10.17.3]). Дээрх томьёог ашиглан k_ε тогтмолыг ε нарийвчлалтай тооцоход

$$k_\varepsilon = -e^{i\frac{\pi}{4}} + (2\sqrt{2})^{-1}\varepsilon e^{-i(\frac{1}{\varepsilon}+\frac{\pi}{4})} + O(\varepsilon^2)$$

гэж олдсон(тооцооллыг хавсралт хэсгээс харж болно).

2.1 Амплитуд функц $m_\varepsilon(t)$

$m_\varepsilon(t)$ амплитуд функцийг олохдоо (I.6) тэгшитгэлийн $l_\varepsilon(t)$-ийн оронд (II.2) орлуулбал:

$$\begin{cases} \dfrac{m'_\varepsilon(t)}{m_\varepsilon(t)} = \dfrac{l_\varepsilon(t)}{2i} = \dfrac{-u'_\varepsilon(t)}{2u_\varepsilon(t)} \\ m_\varepsilon(-1/\varepsilon) = \pi^{-1/4} \end{cases}$$

гэсэн тэгшитгэл үүснэ. Энэ тэгшигэлээс $m_\varepsilon(t) = A_\varepsilon u_\varepsilon^{-1/2}$ хэлбэртэй байхыг амархан харж болно. Энд A_ε тогтмол, $u_\varepsilon(t) = \left(\frac{\tau}{2}\right)^{\frac{1}{2}}\left[-J_{-\frac{1}{4}}\left(\frac{\tau^2}{2}\right) + k_\varepsilon J_{\frac{1}{4}}\left(\frac{\tau^2}{2}\right)\right]$
II.4 томьёоны a тогтмолыг -1 байхаар сонгосон). Одоо амплитуд функц $m_\varepsilon(t)$ анхны нөхцөлийг орлуулбал:

$$\begin{aligned} \pi^{-1/4} &= m_\varepsilon(-1/\varepsilon) = A_\varepsilon u_\varepsilon(-1/\varepsilon)^{-1/2} \\ &= A_\varepsilon\left[\left(\frac{\tau}{2}\right)^{\frac{1}{2}}\left(-J_{-\frac{1}{4}}\left(\frac{\tau^2}{2}\right) - k_\varepsilon J_{\frac{1}{4}}\left(\frac{\tau^2}{2}\right)\right)\right]^{-1/2}\Bigg|_{\tau=\frac{1}{\sqrt{\varepsilon}}} \end{aligned}$$

гэж гарна. Эндээс A_ε тогтмол

$$A_\varepsilon = \frac{\varepsilon^{1/8}}{\sqrt{\pi}}(e^{-i(\frac{1}{2\varepsilon}-\frac{7\pi}{8})} + \frac{\varepsilon}{16}e^{-i(\frac{1}{2\varepsilon}-\frac{3\pi}{8})} - \frac{\varepsilon}{4}e^{-i(\frac{3}{2\varepsilon}-\frac{\pi}{8})} + O(\varepsilon^2))^{1/2}$$

гэж олдоно.

3 $L > 0$ үеийн шийд

$t \to 0$ үед $l_\varepsilon(t)$ болон $m_\varepsilon(t)$ функцүүд харгалзан $l_\varepsilon(t) = ia_\varepsilon \varepsilon^{\frac{1}{2}}(1 + a_\varepsilon \varepsilon^{\frac{1}{2}} t + O(\varepsilon^{\frac{1}{2}} t)^2)$, $m_\varepsilon(t) = -iA_\varepsilon \Gamma(3/4)^{1/2}\Big(1 + \frac{\Gamma(3/4)}{\Gamma(1/4)} k_\varepsilon \sqrt{\varepsilon} t + O(t^2)\Big)$ (тооцооллыг хавсралт хэсгээс харж болно)байна. Энд $a_\varepsilon = 2k_\varepsilon \Gamma(3/4)/\Gamma(1/4)$. $L = 0$ үеийн шийдийг хугацааны хувьд L-р параллель зөөлт хийснээр $L > 0$ үеийн шийдийг гаргаж авах ба $-1/\varepsilon$ дээрх анхны нөхцөл нь буцаад $-(L+1)/\varepsilon$ дээрх анхны нөхцөл болно. Харин l_ε, m_ε функцүүдийн 0 дээрх утгууд $-L/\varepsilon$ дээрх утга болж өөрчлөгдөнө.Үүний дараагаар $m_\varepsilon(-L)$, $l_\varepsilon(-L)$ утгуудыг чөлөөт Шредингерийн тэгшитгэлийн анхны нөхцөлөөр сонгож шийдээ тооцоолно. Энэ нь бидэнд хугацааны $t \in [-(L+1)/\varepsilon, -L/\varepsilon]$ болон $t \in [-L/\varepsilon, L/\varepsilon]$ завсар дахь шийдүүдийг нийлүүлж өгнө. Өөрөөр хэлбэл бид хугацааны $t \in [-(L+1)/\varepsilon, L]$ завсарт шийдийг олсон гэсэн үг. Уг гаргаж авсан шийдийн $t = L/\varepsilon$ агшин дахь утгыг ашиглан хугацааны $t \in [L/\varepsilon, (L+1)/\varepsilon]$ завсар дахь шийдийн анхны нөхцөлөөр өгч хугацааны $t \in [-(L+1)/\varepsilon, (L+1)/\varepsilon]$ завсар дахь буюу (I.2) тэгшитгэлийн шийдийг олно.

3.1 Чөлөөт Шредингерийн тэгшитгэлийн шийд

$i\varepsilon\partial_t u_\varepsilon = Hu$, $H = -\frac{\hbar}{2m}\Delta$ ба анхны нөхцөлийн функц нь u_0 байг. Тэгвэл энэ тэгшитгэлийн шийд $u(t) := U(t)u_0$ ба $U(t) := e^{-iHt/\hbar}$ байдаг ([3, 2.4] хэсгээс харж болно). Эндээс $t = L/\varepsilon$ үеийн утгыг тооцвол

$$\begin{aligned} v_\varepsilon(L/\varepsilon, x) &= \frac{-iA_\varepsilon \Gamma(3/4)^{\frac{1}{2}} e^{-i\frac{\pi}{4}}}{(4\pi L/\varepsilon)^{\frac{1}{2}}} \int_{\mathbb{R}} e^{\frac{i\varepsilon(x-y)^2}{4L} - \frac{ia_\varepsilon\sqrt{\varepsilon}y^2}{2}} dy \\ &= \frac{-iA_\varepsilon \Gamma(3/4)^{\frac{1}{2}} \varepsilon^{\frac{1}{4}} e^{-i\frac{\pi}{2}}}{(\sqrt{\varepsilon} - 2La_\varepsilon)^{\frac{1}{2}}} e^{-\frac{i\varepsilon a_\varepsilon}{2(\sqrt{\varepsilon}-2La_\varepsilon)}x^2} \end{aligned}$$

гэж гарна.

3.2 $t \in (L/\varepsilon, (L+1)/\varepsilon)$ завсар дахь шийд

Өмнөх хэсэгт бид Шредингерийн тэгшитгэлийн шийдийг $L > 0$ үед хугацааны $t \in (-(L+1)/\varepsilon, L/\varepsilon)$ завсарт байгуулсан. Одоо бид (III.2) системээр өгөгдөх тэгшитгэлийг бодно. Энэхүү тэгшитгэлийн шийд бидэнд (I.2) тэгшитгэлийн шийдийг өгнө.

$$\begin{cases} \partial_t v_\varepsilon(t,x) = -\frac{1}{2}\Delta v_\varepsilon + \frac{(t-L/\varepsilon)^2\varepsilon^2 x^2}{2} v_\varepsilon \\ v_\varepsilon(L/\varepsilon, x) = \dfrac{-iA_\varepsilon \Gamma(3/4)^{\frac{1}{2}} \varepsilon^{\frac{1}{4}} e^{-i\frac{\pi}{2}}}{(\sqrt{\varepsilon} - 2La_\varepsilon)^{\frac{1}{2}}} e^{-\frac{i\varepsilon a_\varepsilon}{2(\sqrt{\varepsilon}-2La_\varepsilon)}x^2} \end{cases}$$

Хэрэв бид $\tau = t - L/\varepsilon$ орлуулга хийвэл дээрх тэгшитгэл шийд $L = 0$ үед тооцсон $l_\varepsilon(t),\ m_\varepsilon(t)$ функцүүдтэй адилхан хэлбэрээр илэрхийлэгдэнэ.

Өөрөөр хэлбэл

$$\begin{cases} l'_\varepsilon(\tau) + il_\varepsilon(\tau)^2 = i(\varepsilon\tau)^2 \\ l_\varepsilon(0) = \frac{i\varepsilon a_\varepsilon}{\sqrt{\varepsilon}-2La_\varepsilon} \end{cases} \tag{3.1}$$

болон

$$\begin{cases} im_\varepsilon(\tau) = \frac{1}{2}m_\varepsilon(\tau)l_\varepsilon(\tau), \\ m_\varepsilon(0) = \frac{-iA_\varepsilon\Gamma(3/4)^{\frac{1}{2}}\varepsilon^{\frac{1}{4}}}{(\sqrt{\varepsilon}-2La_\varepsilon)^{\frac{1}{2}}} \end{cases} \tag{3.2}$$

тэгшитгэлүүд үүснэ.

(III.1), (III.2) тэгшитгэлийн шийдийн ерөнхий хэлбэр нь ($L = 0$ үед тооцсон):

$$l_\varepsilon(\tau, \gamma) = \frac{-i\sqrt{\varepsilon}\tau\left[J_{\frac{3}{4}}\left(\frac{\tau^2}{2}\right) + \gamma J_{-\frac{3}{4}}\left(\frac{\tau^2}{2}\right)\right]}{-J_{-\frac{1}{4}}\left(\frac{\tau^2}{2}\right) + \gamma J_{\frac{1}{4}}\left(\frac{\tau^2}{2}\right)}$$

$$m_\varepsilon(\tau) = B_\varepsilon\left[\left(\frac{\tau}{2}\right)^{\frac{1}{2}}\left(-J_{-\frac{1}{4}}\left(\frac{\tau^2}{2}\right) + \gamma J_{\frac{1}{4}}\left(\frac{\tau^2}{2}\right)\right)\right]^{-1/2}$$

байна. Анхны нөхцөлөөс γ, B_ε коеффициентүүдийг тооцоход

$$\gamma = \frac{\varepsilon^{\frac{1}{2}}k_\varepsilon}{\sqrt{\varepsilon} - 2La_\varepsilon}, \quad B_\varepsilon = \frac{A_\varepsilon\varepsilon^{\frac{1}{4}}}{(\sqrt{\varepsilon} - 2La_\varepsilon)^{\frac{1}{2}}} \text{ гэж гарсан.}$$

$\tau = 1/\varepsilon$ байх үеийн (t-ийн хувьд $t = (L+1)/\varepsilon$) l_ε, m_ε функцүүдийн утгыг сонирхъё. $C = \Gamma(3/4)/\Gamma(1/4)$ гэсэн тэмдэглэгээ хийе. Тэгвэл

$$l_\varepsilon(1/\varepsilon) = i\frac{-4k_\varepsilon CLJ_{\frac{3}{4}}(\frac{\tau^2}{2}) + \sqrt{\varepsilon}\big(k_\varepsilon J_{-\frac{3}{4}}(\frac{\tau^2}{2}) + J_{\frac{3}{4}}(\frac{\tau^2}{2})\big)}{-4k_\varepsilon CLJ_{-\frac{1}{4}}(\frac{\tau^2}{2}) - \sqrt{\varepsilon}\big(k_\varepsilon J_{\frac{1}{4}}(\frac{\tau^2}{2}) - J_{-\frac{1}{4}}(\frac{\tau^2}{2})\big)}\Bigg|_{\frac{1}{\sqrt{\varepsilon}}} \tag{3.3}$$

$$m_\varepsilon(1/\varepsilon) = B_\varepsilon\left[\left(\frac{\tau}{2}\right)^{\frac{1}{2}}\left(-J_{-\frac{1}{4}}\left(\frac{\tau^2}{2}\right) + \gamma J_{\frac{1}{4}}\left(\frac{\tau^2}{2}\right)\right)\right]^{-\frac{1}{2}}\Bigg|_{\frac{1}{\sqrt{\varepsilon}}} \tag{3.4}$$

байна. $L_1 = -4CL$ тэмдэглэгээ хийж хугацааны L/ε параллель зөөлт хийвэл (III.3) тэгшитгэл:

$$l_\varepsilon((L+1)/\varepsilon) = i\frac{k_\varepsilon L_1 J_{\frac{3}{4}}(\frac{\tau^2}{2}) + \sqrt{\varepsilon}\big(k_\varepsilon J_{-\frac{3}{4}}(\frac{\tau^2}{2}) + J_{\frac{3}{4}}(\frac{\tau^2}{2})\big)}{k_\varepsilon L_1 J_{-\frac{1}{4}}(\frac{\tau^2}{2}) - \sqrt{\varepsilon}\big(k_\varepsilon J_{\frac{1}{4}}(\frac{\tau^2}{2}) - J_{-\frac{1}{4}}(\frac{\tau^2}{2})\big)} \tag{3.5}$$

Харин (III.4) тэгшитгэл:

$$\begin{aligned} m_\varepsilon((L+1)/\varepsilon)^2 &= B_\varepsilon^2\left[\left(\frac{\tau}{2}\right)^{\frac{1}{2}}\left(-J_{-\frac{1}{4}}\left(\frac{\tau^2}{2}\right) + \gamma J_{\frac{1}{4}}\left(\frac{\tau^2}{2}\right)\right)\right]^{-1}\Bigg|_{\tau=\frac{1}{\sqrt{\varepsilon}}} \\ &= \frac{-\varepsilon\sqrt{2}\pi^{-1}e^{-i\left(\frac{1}{2\varepsilon}-\frac{7\pi}{8}\right)}}{L_1 k_\varepsilon J_{-\frac{1}{4}}\left(\frac{1}{2\varepsilon}\right) - \sqrt{\varepsilon}\left(k_\varepsilon J_{\frac{1}{4}}\left(\frac{1}{2\varepsilon}\right) - J_{-\frac{1}{4}}\left(\frac{1}{2\varepsilon}\right)\right)} \end{aligned} \tag{3.6}$$

болно. (III.5) томьёогоор илэрхийлэгдэх $l_\varepsilon((L+1)/\varepsilon)$ функцийг комплекс хосмогоор нь үржиж (II.5) томьёог ашиглан ε нарийвчлалтай тооцоо хийвэл:

$$\begin{aligned} l_\varepsilon((L+1)/\varepsilon) &= (8\varepsilon + i(8B^2\sin 2t + 8\sqrt{2\varepsilon}B\cos 2t \\ &+ 2L_1^2\varepsilon(3 - \cos 4t + \sin 4t))) : (8\varepsilon + 8B^2 \\ &+ 8B^2\cos 2t - 8\sqrt{2\varepsilon}B\sin 2t + L_1^2\varepsilon(2 \\ &+ 4\cos 2t + 2\cos 4t + 7\sin 2t + 2\sin 4t)) \end{aligned}$$

Энд $t = \frac{1}{2\varepsilon} - \frac{\pi}{8}$, $B = L_1 - \sqrt{2\varepsilon}$.

Теорем 3.1. *$\varepsilon \to 0$ үед $l_\varepsilon((L+1)/\varepsilon)$ функцийн бодит хэсэг $0 < \Re l_\varepsilon((L+1)/\varepsilon) \le 1$ хооронд $|\sin(1/\varepsilon)|$ төрлийн хэлбэлзэл хийнэ.*

Харин хуурмаг хэсэг $-\infty < \Im l_\varepsilon((L+1)/\varepsilon) < \infty$ *хооронд* $\cot(1/\varepsilon)$ *төрлийн хэлбэлзэл хийнэ.*

Баталгаа. $\frac{1}{\varepsilon} = \tau,\ t = \frac{1}{2\varepsilon} - \frac{\pi}{8} = \frac{\tau}{2} - \frac{\pi}{8}$, тэмдэглэгээ хийе.

$$
\begin{aligned}
g(\tau) &= 8\left(L_1 - \sqrt{\frac{2}{\tau}}\right)^2 \left[1 + \cos\left(\tau - \frac{\pi}{4}\right)\right] \\
&- 8\sqrt{\frac{2}{\tau}}\left(L_1 - \sqrt{\frac{2}{\tau}}\right)\sin\left(\tau - \frac{\pi}{4}\right) + \frac{8}{\tau} \\
&+ \frac{L_1^2}{\tau}\left[2 + 4\cos\left(\tau - \frac{\pi}{4}\right) + 2\cos 2\left(\tau - \frac{\pi}{4}\right)\right. \\
&+ \left. 7\sin\left(\tau - \frac{\pi}{4}\right) + 2\sin 2\left(\tau - \frac{\pi}{4}\right)\right]
\end{aligned}
$$

$f(\tau) = \tau g(\tau)$

$$
\begin{aligned}
f(\tau) &= 8\left(L_1\sqrt{\tau} - \sqrt{2}\right)^2 \left[1 + \cos\left(\tau - \frac{\pi}{4}\right)\right] \\
&- 8\sqrt{2}\left(L_1\sqrt{\tau} - \sqrt{2}\right)\sin\left(\tau - \frac{\pi}{4}\right) + 8 \\
&+ L_1^2\left[2 + 4\cos\left(\tau - \frac{\pi}{4}\right) + 2\cos 2\left(\tau - \frac{\pi}{4}\right)\right. \\
&+ \left. 7\sin\left(\tau - \frac{\pi}{4}\right) + 2\sin 2\left(\tau - \frac{\pi}{4}\right)\right]
\end{aligned}
$$

$f(\tau)$ функцийн синус, косинус функцүүдийг $\tau_0 = \frac{\pi}{4} + \pi + 2\pi k,\ k \in \mathbb{N}$ цэгийн орчинд Тейлорын цуваанд задалъя $\xi_i = \alpha\tau + (1-\alpha)\tau_0$, $0 < \alpha < 1$, $i = 1, 2, \ldots, 6$.

$$
\begin{aligned}
f(\tau) &= 8\left(L_1\sqrt{\tau} - \sqrt{2}\right)^2 \frac{-\cos\xi_1}{2}(\tau - \tau_0)^2 \\
- &\quad 8\sqrt{2}\left(L_1\sqrt{\tau} - \sqrt{2}\right)\cos\xi_2(\tau - \tau_0) \\
+ &\quad 8 + L_1^2\ (2 + 4\left(-1 - \frac{\cos\xi_3}{2}(\tau - \tau_0)^2\right) \\
+ &\quad 2[1 - 2\cos 2\xi_4(\tau - \tau_0)^2] + 7\cos\xi_5 \\
\cdot &\quad (\tau - \tau_0) + 4\cos 2\xi_6(\tau - \tau_0))\ \text{байна.}
\end{aligned}
$$

Дээрх задаргааг эмхтгэвэл:

$$\begin{aligned}
f(\tau) &= -8\left(L_1\sqrt{\tau}-\sqrt{2}\right)^2\frac{\cos\xi_1}{2}(\tau-\tau_0)^2 \\
- &\quad 8\sqrt{2}\left(L_1\sqrt{\tau}-\sqrt{2}\right)\cos\xi_2(\tau-\tau_0)+8 \\
+ &\quad L_1^2\,(-\frac{\cos\xi_3}{2}(\tau-\tau_0)^2-4[\cos 2\xi_4(\tau-\tau_0)^2] \\
+ &\quad 7\cos\xi_5\,(\tau-\tau_0)+4\cos 2\xi_6(\tau-\tau_0)) \\
= &\quad -4\left(L_1\sqrt{\tau}-\sqrt{2}\right)^2(\tau-\tau_0)^2\cos\xi_1 \\
- &\quad 8\sqrt{2}\left(L_1\sqrt{\tau}-\sqrt{2}\right)(\tau-\tau_0)\cos\xi_2+8 \\
+ &\quad L_1^2\,(-\frac{\cos\xi_3}{2}(\tau-\tau_0)^2-4(\tau-\tau_0)^2\cos 2\xi_4 \\
+ &\quad 7(\tau-\tau_0)\cos\xi_5+4(\tau-\tau_0)\cos 2\xi_6)\ \text{болно.}
\end{aligned}$$

$\tau=\tau_0$ цэгийн орчинд косинус функц үргэлж сөрөг утга авна. Иймд τ зэргээр гол гишүүнээ сонговол:

$$-4\left(L_1\sqrt{\tau}-\sqrt{2}\right)^2(\tau-\tau_0)^2\cos\xi_1>0 \text{ байна.}$$

Хэрэв k хангалттай том утга авч $|\tau-\tau_0|<\frac{\pi}{8}$ байх үед $f(\tau)\geq 8$ байна. Харин $|\tau-\tau_0|>\frac{\pi}{8}$ үед

$$8\left(L_1\sqrt{\tau}-\sqrt{2}\right)^2\left[1+\cos\left(\tau-\frac{\pi}{4}\right)\right]$$

гол гишүүн болж $\tau\to\infty$ үед $f(\tau)\to\infty$ байна. Эндээс $\varepsilon\to 0$ үед $l((L+1)/\varepsilon)$ функцийн бодит хэсэг үргэлж эерэг утгатай ба $\lim\limits_{\tau\to\tau_0}\frac{8}{f(\tau)}=1$ байна.

Одоо хуурмаг хэсгийн тооцог хийе.

$\tau\to\infty$ үед хуурмаг хэсэг $\Im l_\varepsilon=\frac{\sin(\tau-\frac{\pi}{4})}{1+\cos(t-\frac{\pi}{4})}$ болохыг амархан харж болно. $\tau\to\pi+\frac{\pi}{4}+2\pi k$ үеийн хязгаарыг Лопиталийн дүрэм ашиглан тооцвол:

$$\lim_{\tau\to\pi+\frac{\pi}{4}+2\pi k}\Im l_\varepsilon=\lim_{\tau\to\pi+\frac{\pi}{4}+2\pi k}\cot\left(\tau-\frac{\pi}{4}\right)$$

$\tau \to \pi + \frac{\pi}{4} + 2\pi k$ үеийн $\cot(\tau - \pi/4)$ функцийн зүүн өрөөсгөл хязгаар нь ∞, баруун өрөөсгөл хязгаар нь $-\infty$. Иймд хуурмаг хэсэг $(-\infty, \infty)$ хооронд хэлбэлзэнэ. $\square$

Дээрх теоремоос дараах мөрдлөгөө хэлж болно.

Мөрдлөгөө 3.1. *Хангалттай том k, $k \in \mathbb{N}$ -ийн утгад $\varepsilon = \frac{4}{5\pi + 8\pi k}$ цэгүүд дээр $\Re l_\varepsilon((L+1)/\varepsilon) = 1$ байна.*

Теорем 3.2. *$\varepsilon \to 0$ үед (I.2) тэгшитгэлээр өгөгдөх Шредингерийн тэгшитгэлийн шийд $v_\varepsilon(t,x)$ функцийн $t = (L+1)/\varepsilon$ агшин дахь утга анхны нөхцөл $v_\varepsilon(-(L+1/\varepsilon)) = \pi^{-\frac{1}{4}} e^{-\frac{x^2}{2}}$ функц дээрх проекц 0 буюу анхны нөхцөлийн функцтэй ортогонал байна.*

Баталгаа.

$$\begin{aligned} & \langle v_\varepsilon((L+1)/\varepsilon), v_\varepsilon(-(L+1)/\varepsilon\rangle \\ = & \int_{\mathbb{R}} \pi^{-\frac{1}{4}} e^{-\frac{x^2}{2}} m_\varepsilon((L+1)/\varepsilon) e^{-l_\varepsilon((L+1)/\varepsilon)\frac{x^2}{2}} dx \\ = & \frac{\sqrt{2}\pi^{\frac{1}{4}} m_\varepsilon((L+1)/\varepsilon)}{(1 + l_\varepsilon((L+1)/\varepsilon))^{\frac{1}{2}}} \text{ болно.} \end{aligned}$$

$$P_\varepsilon((L+1)/\varepsilon) = ik_\varepsilon L_1 J_{\frac{3}{4}}(\frac{\tau^2}{2}) + \sqrt{\varepsilon}\big(k_\varepsilon J_{-\frac{3}{4}}(\frac{\tau^2}{2}) + J_{\frac{3}{4}}(\frac{\tau^2}{2})\big)\Big|_{\tau = \frac{1}{\sqrt{\varepsilon}}}$$
$$Q_\varepsilon((L+1)/\varepsilon) = k_\varepsilon L_1 J_{-\frac{1}{4}}(\frac{\tau^2}{2}) - \sqrt{\varepsilon}\big(k_\varepsilon J_{\frac{1}{4}}(\frac{\tau^2}{2}) - J_{-\frac{1}{4}}(\frac{\tau^2}{2})\big)\Big|_{\tau = \frac{1}{\sqrt{\varepsilon}}}$$

гэсэн тэмдэглэгээ хийж (III.6) томьёог орлуулбал:

$$\begin{aligned} |\langle v_\varepsilon((L+1)/\varepsilon), v_\varepsilon(-(L+1)/\varepsilon)\rangle|^2 &= \tfrac{2\pi^{\frac{1}{2}} m_\varepsilon((L+1)/\varepsilon)^2}{1 + l_\varepsilon((L+1)/\varepsilon)} \\ &= \tfrac{-2\sqrt{2}\pi^{-\frac{1}{2}} \varepsilon e^{-i\left(\frac{1}{2\varepsilon} - \frac{7\pi}{8}\right)} (1 + O(\varepsilon))}{\left(P_\varepsilon((L+1)/\varepsilon) + Q_\varepsilon((L+1)/\varepsilon)\right)}. \end{aligned} \tag{3.7}$$

(III.7) тэгшитгэлийн хуваарийг тооцъё.

$$\begin{aligned} & P_\varepsilon((L+1)/\varepsilon) + Q_\varepsilon((L+1)/\varepsilon) = \\ = \quad & L_1 k_\varepsilon J_{-\frac{1}{4}}\left(\frac{1}{2\varepsilon}\right) - \sqrt{\varepsilon}\left(k_\varepsilon J_{\frac{1}{4}}\left(\frac{1}{2\varepsilon}\right) - J_{-\frac{1}{4}}\left(\frac{1}{2\varepsilon}\right)\right) \\ + \quad & i k_\varepsilon L_1 J_{\frac{3}{4}}\left(\frac{1}{2\varepsilon}\right) + i\sqrt{\varepsilon}\left(k_\varepsilon J_{-\frac{3}{4}}\left(\frac{1}{2\varepsilon}\right) + J_{\frac{3}{4}}\left(\frac{1}{2\varepsilon}\right)\right) \\ = \quad & -L_1 e^{i\frac{\pi}{4}}\left(\cos(\frac{1}{2\varepsilon} - \frac{\pi}{8}) + i\sin(\frac{1}{2\varepsilon} - \frac{\pi}{8})\right) + O(\sqrt{\varepsilon}) \end{aligned}$$

Дээрх үр дүнг (III.7) тэгшитгэлд орлуулбал:

$$\begin{aligned} & |\langle v_\varepsilon((L+1)/\varepsilon), v_\varepsilon(-(L+1)/\varepsilon)\rangle|^2 = \\ = \quad & \frac{-2\sqrt{2}\pi^{-\frac{1}{2}}\varepsilon e^{-i\left(\frac{1}{2\varepsilon} - \frac{7\pi}{8}\right)}(1 + O(\varepsilon))}{-L_1 e^{i\frac{\pi}{4}}(\cos(\frac{1}{2\varepsilon} - \frac{\pi}{8}) + i\sin(\frac{1}{2\varepsilon} - \frac{\pi}{8})) + O(\sqrt{\varepsilon})} \\ = \quad & -2\sqrt{2}L_1^{-1}\pi^{-\frac{1}{2}}\varepsilon e^{-i\left(\frac{1}{\varepsilon} + \frac{\pi}{4}\right)}\left(1 + O(\varepsilon\sqrt{\varepsilon})\right) \end{aligned}$$

$\varepsilon \to 0$ үед $|\langle v_\varepsilon((L+1)/\varepsilon), v_\varepsilon(-(L+1)/\varepsilon)\rangle|^2 \to 0$. □

Ашигласан ном

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4 Хавсралт

Асимптот задаргаа

$a = x - \frac{1}{2}\nu\pi - \frac{\pi}{4}$

$$J_{-\frac{1}{4}}J_{\frac{1}{4}} = \frac{1}{\pi x}\Big(\cos 2a + \cos\frac{\pi}{4} + \frac{3}{16x}\sin 2a + O\Big(\frac{1}{x^2}\Big)\Big)$$

$$J_{\frac{3}{4}}J_{-\frac{3}{4}} = \frac{1}{\pi x}\Big(\cos 2a + \cos\frac{3\pi}{4} - \frac{5}{16x}\sin 2a + O\Big(\frac{1}{x^2}\Big)\Big)$$

$$J^2_{\frac{1}{4}} = \frac{2}{\pi x}\Big(\cos^2(a - \frac{\pi}{8}) + \frac{3}{32x}\sin 2(a - \frac{\pi}{8}) + O\Big(\frac{1}{x^2}\Big)\Big)$$

$$J^2_{-\frac{3}{4}} = \frac{2}{\pi x}\Big(\cos^2(a + \frac{3\pi}{8}) - \frac{5}{32x}\sin 2(a + \frac{3\pi}{8}) + O\Big(\frac{1}{x^2}\Big)\Big)$$

$$J^2_{\frac{3}{4}} = \frac{2}{\pi x}\Big(\sin^2(a + \frac{\pi}{8}) + \frac{5}{32x}\sin(2a + \frac{\pi}{4}) + O\Big(\frac{1}{x^2}\Big)\Big)$$

$$J^2_{-\frac{1}{4}} = \frac{2}{\pi x}\Big(\cos^2(a + \frac{\pi}{8}) + \frac{3}{32x}\sin(2a + \frac{\pi}{4}) + O\Big(\frac{1}{x^2}\Big)\Big)$$

$$
\begin{aligned}
J_{\frac{3}{4}}J_{\frac{1}{4}} &= \frac{2}{\pi x}\Big(\sin(a+\frac{3\pi}{8})\cos(a-\frac{3\pi}{8})\\
&- \frac{5}{32x}\sin(a-\frac{3\pi}{8})\sin(a+\frac{3\pi}{8})\\
&- \frac{3}{32x}\cos(a-\frac{3\pi}{8})\cos(a+\frac{3\pi}{8})+O\Big(\frac{1}{x^2}\Big)\Big)
\end{aligned}
$$

$$
\begin{aligned}
J_{-\frac{3}{4}}J_{-\frac{1}{4}} &= \frac{2}{\pi x}\Big(-\sin(a-\frac{3\pi}{8})\cos(a+\frac{3\pi}{8})\\
&+ \frac{5}{32x}\sin(a-\frac{3\pi}{8})\sin(a+\frac{3\pi}{8})\\
&+ \frac{3}{32x}\cos(a-\frac{3\pi}{8})\cos(a+\frac{3\pi}{8})-O\Big(\frac{1}{x^2}\Big)\Big)
\end{aligned}
$$

$$
J_{\frac{3}{4}}J_{-\frac{1}{4}} = \frac{1}{\pi x}\Big(\frac{4}{16x}+\frac{\cos(2a+\frac{\pi}{4})}{16x}+\sin(2a+\frac{\pi}{4})+O\Big(\frac{1}{x^2}\Big)\Big)
$$

$$
\begin{aligned}
k_\varepsilon &= -\left.\frac{J_{-\frac{1}{4}}+iJ_{\frac{3}{4}}}{J_{\frac{1}{4}}-iJ_{-\frac{3}{4}}}\right|_{\frac{1}{2\varepsilon}}\\
&= -\left.\frac{e^{i\frac{\pi}{4}}+\frac{\sin 2a}{2x}+O\Big(\frac{1}{x^2}\Big)}{1+\frac{\sin(2a-\frac{\pi}{4})}{4x}+O\Big(\frac{1}{x^2}\Big)}\right|_{\frac{1}{2\varepsilon}}\\
&= -e^{i\frac{\pi}{4}}+\varepsilon\frac{e^{-i(\frac{1}{\varepsilon}+\frac{\pi}{4})}}{2\sqrt{2}}+O(\varepsilon^2)
\end{aligned}
$$

$$
\begin{aligned}
A_\varepsilon &= \pi^{-1/4}u_\varepsilon(-1/\varepsilon)^{1/2}\\
&= \pi^{-1/4}\left.\left[\left(\frac{\tau}{2}\right)^{\frac{1}{2}}\left(-J_{-\frac{1}{4}}\left(\frac{\tau^2}{2}\right)-k_\varepsilon J_{\frac{1}{4}}\left(\frac{\tau^2}{2}\right)\right)\right]^{1/2}\right|_{\tau=\frac{1}{\sqrt{\varepsilon}}}\\
&= \frac{\varepsilon^{1/8}}{\sqrt{\pi}}(e^{-i(\frac{1}{2\varepsilon}-\frac{7\pi}{8})}+\frac{\varepsilon}{16}e^{-i(\frac{1}{2\varepsilon}-\frac{3\pi}{8})}-\frac{\varepsilon}{4}e^{-i(\frac{3}{2\varepsilon}-\frac{\pi}{8})}+O(\varepsilon^2))^{1/2}
\end{aligned}
$$

$t \to 0$ байх үеийн m_ε функцийн хэлбэр:

$$
\begin{aligned}
m_\varepsilon(t) &= \frac{-iA_\varepsilon}{(\Gamma(3/4)^{-1}-\frac{k_\varepsilon\sqrt{\varepsilon}t}{2\Gamma(4/5)}+)^{1/2}}\\
&= \frac{-iA_\varepsilon\Gamma(3/4)^{1/2}}{(1-\frac{2\Gamma(3/4)}{\Gamma(1/4)}k_\varepsilon\sqrt{\varepsilon}t)^{1/2}}\\
&= -iA_\varepsilon\Gamma(3/4)^{1/2}\Big(1+\frac{\Gamma(3/4)}{\Gamma(1/4)}k_\varepsilon\sqrt{\varepsilon}t+O(t^2)\Big)
\end{aligned}
$$

$t \to 0$ байх үеийн l_ε функцийн хэлбэр:

$$\begin{aligned}
l_\varepsilon(t) &= \\
&- \quad 2i\sqrt{\varepsilon}\left(-a\sum_{k=0}^{\infty}\frac{(-1)^k(\tau/2)^{4k+3}}{k!\Gamma(k+\frac{3}{4}+1)} + b\sum_{k=0}^{\infty}\frac{(-1)^k(\tau/2)^{4k}}{k!\Gamma(k-\frac{3}{4}+1)}\right) \\
&: \quad \left(a\sum_{k=0}^{\infty}\frac{(-1)^k(\tau/2)^{4k}}{k!\Gamma(k-\frac{1}{4}+1)} + b\sum_{k=0}^{\infty}\frac{(-1)^k(\tau/2)^{4k+1}}{k!\Gamma(k+\frac{1}{4}+1)}\right) \\
&= \quad \frac{4k_\varepsilon\sqrt{\varepsilon}\Gamma(1/4)^{-1}}{-2\Gamma(3/4)^{-1} + k_\varepsilon\sqrt{\varepsilon}t\Gamma(5/4)^{-1}} \\
&= \quad \frac{ia_\varepsilon\sqrt{\varepsilon}}{1-a_\varepsilon\sqrt{\varepsilon}t} = ia_\varepsilon\varepsilon^{\frac{1}{2}}\left(1 + a_\varepsilon\varepsilon^{\frac{1}{2}}t + \left(a_\varepsilon\varepsilon^{\frac{1}{2}}t\right)^2 + O\left(\varepsilon^{\frac{1}{2}}t\right)^3\right)
\end{aligned}$$

Энд $a_\varepsilon = 2k_\varepsilon\Gamma(3/4)/\Gamma(1/4)$

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Non-relativistic Pauli–Fierz Hamiltonian for less than two photons

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Abstract. We consider the Pauli–Fierz model, which describes a particle (an electron) coupled to the quantized electromagnetic field and limit the number of photons to less than 2. By computing the resolvent explicitly, we located the spectrum of the Hamiltonian mass. Our results do not depend on the coupling constant e nor on the infrared cutoff parameter R.

Key words: Pauli–Fierz Hamiltonian, mass renormalization, dressed electron states.

1. Introduction

In this paper, we study the fiber Hamiltonian for the standard model of non-relativistic quantum electrodynamics, called the Pauli–Fierz model, when the number of photons is restricted to 0 or 1. The latter condition allows us to compute the resolvent of the fiber Hamiltonian explicitly. Therefore, the spectrum and the effective mass can be obtained for arbitrary values of parameters such as the coupling constant with the electromagnetic field and the ultraviolet cutoff radius.

There is a rich literature on the spectral and scattering properties of this model. The spectral properties of the Pauli-Fierz Hamiltonian was studied in [5] and the existence of the ground states of the Pauli-Fierz Hamiltonian was proved in [6]. See [11] for discussions on the spectral properties and scattering theory of the Nelson Hamiltonian. The spin-boson model and the Nelson model are discussed in [13] and [4] for when the number of photons is restricted to a few. This work was initially inspired by the paper [14], where the author considered a model with less than two phonons *without polarization* and computed the spectrum, the ground state, as well as the effective mass.

An extensive review of the properties of the ground state of the fiber

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Hamiltonian, its differentiability, and the effective mass can be found in [1]. Most of the previous results were derived for various conditions for the above-mentioned parameters and for a certain limited range of the total momentum. We note that for the Nelson model, the spectrum shifts from zero to negative values due to the radiation field, which is not observed in our setup.

Let us introduce the model. We set the bare electron mass m and the speed of light c to be equal to 1. The Hilbert space for the system is given by $\mathcal{H} := L^2(\mathbb{R}^3_x) \otimes \mathcal{F}$, where the bosonic Fock space $\mathcal{F}$ is defined by

$$\mathcal{F} = \bigoplus_{n=0}^{\infty} \mathcal{F}^{(n)} = \bigoplus_{n=0}^{\infty} \left(\otimes_s^n \mathcal{S}\right),$$

with $\mathcal{S} = L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$. We have denoted the n-fold symmetric tensor product of $\mathcal{S}$ by $\otimes_s^n \mathcal{S}$, with $\otimes_s^0 \mathcal{S} = \mathbb{C}$. The annihilation and the creation operators a and a^* are defined as

$$a^\sharp(v) = \sum_{\lambda=1}^{2} \int a^\sharp(k,\lambda) v(k,\lambda) dk, \tag{1}$$

for $v = (v(\cdot,1), v(\cdot,2)) \in L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$, where $a^\sharp$ is for a or a^* and $a^\sharp(k,\lambda)$ is a formal kernel. The free photon field operator H_f on $\mathcal{F}$ is defined as

$$H_f = \sum_{\lambda=1}^{2} \int \omega(k) a^*(k,\lambda) a(k,\lambda) dk,$$

where $\omega(k) = |k|$ is the photon energy. The quantized radiation field $A_g(x) = (A_{g1}(x), A_{g2}(x), A_{g3}(x))$, $x \in \mathbb{R}^3$, acting on $\mathcal{F}$ is given by

$$A_{gj}(x) = \frac{1}{\sqrt{2}} \sum_{\lambda=1}^{2} \int e_j(k,\lambda) \left[g(k) e^{-ik\cdot x} a^*(k,\lambda) + \overline{g(k)} e^{ik\cdot x} a(k,\lambda) \right] dk.$$

Here, $e(k,\lambda) = (e_1(k,\lambda), e_2(k,\lambda), e_3(k,\lambda))$ are polarization vectors satisfying the conditions $k \cdot e(k,\lambda) = 0$ and $e(k,\lambda) \cdot e(k,\mu) = \delta_{\lambda\mu}$, $\lambda, \mu = 1, 2$.

Assumption We assume that

$$g(k) = \frac{\chi(k)}{\omega^{1/2-\sigma}(k)}, \tag{2}$$

where $\chi(k)$ is a characteristic function of region $\{k \in \mathbb{R}^3 \mid |k| \leq R\}$, $R > 0$ is the ultraviolet cutoff radius, and $0 \leq \sigma < 1/2$ is an infrared renormalization parameter.

Note that the infrared renormalization was introduced only to remove singularities of some auxiliary integrals that appear later. Our main results hold for all values of σ, including zero.

Finally, the Pauli–Fierz Hamiltonian is defined as

$$H = \frac{1}{2}(-i\nabla_x \otimes 1 - eA_g(x))^2 + 1 \otimes H_f, \tag{3}$$

where e is the charge of the electron or the coupling constant with the field. For assumption (2), it was proved in [7] that H is self-adjoint on the domain $D(-(1/2)\Delta + H_f)$ for arbitrary values of the coupling constant e. We define the total momentum operator on $\mathcal{H}$ as

$$P_{\text{tot}} = -i\nabla_x \otimes 1 + 1 \otimes P_f, \tag{4}$$

where $P_f = \sum_{\lambda=1}^{2} \int ka^*(k,\lambda)a(k,\lambda)dk$ is the photon momentum. Since H commutes with P_{tot}, it can be decomposed with respect to the spectrum of P_{tot}:

$$H = \int_{\mathbb{R}^3}^{\oplus} \bar{H}(p)dp,$$

where $\bar{H}(p)$ is defined as

$$\bar{H}(p) = \frac{1}{2}(p - P_f - eA_g(0))^2 + H_f$$

on $\mathcal{F}$. Now $p \in \mathbb{R}^3$ is considered as a parameter. Let E_p be the projection operator onto the 0 or 1 photon space. We introduce the corresponding Pauli–Fierz operator:

$$H(p) := E_p\bar{H}(p)E_p. \tag{5}$$

In the following, we will work only with the operator $H(p)$.

The rest of this paper is organized as follows. We state our main results in Section 2 and compute the resolvent of $H(p)$ in Section 3. Then in Section 4, we locate the spectrum of $H(p)$, and finally in Section 5, we prove our main results.

2. Main results

We are interested in the spectral properties of the Pauli–Fierz Hamiltonian, and in particular, in finding the resolvent of the corresponding fiber Hamiltonian and calculating the effective mass.

The fiber Hamiltonian $H(p)$, defined in the 0 or 1 photon space, is a finite-rank perturbation of an operator whose spectrum consists only of an absolutely continuous part. Therefore, it is very similar to the Friedrichs Hamiltonian. For the spectral characterization of the Friedrichs Hamiltonian and related results, see [10] and [9].

Restricting the number of photons allows us to express the resolvent explicitly. Therefore, all our results were obtained in a nonperturbative way and are independent of parameters such as e and R. In addition, this enables us to work on the scattering properties of this model, as in [4], which, however, will be discussed elsewhere.

We are now ready to formulate the main results. For any $p \in \mathbb{R}^3$, let

$$z_0(|p|) = \min_{k\in\mathbb{R}^3} \left\{ \frac{1}{2}(p-k)^2 + |k| + \gamma_0 \right\} = \begin{cases} \frac{1}{2}p^2 + \gamma_0, & \text{if } |p| \le 1, \\ |p| - \frac{1}{2} + \gamma_0, & \text{if } |p| > 1, \end{cases} \tag{6}$$

be the curve on the $(|p|, z)$ plane, where γ_0 is a constant depending on e and R. An example to keep in mind is

$$\gamma_0 = \frac{\pi}{1+\sigma} e^2 R^{2+2\sigma}.$$

We define the function $F(p, z)$ as

$$F(p,z) = \frac{1}{2}p^2 - z + \gamma_0 \\ - \pi e^2 \left(\frac{1}{2}p^2 + z - \gamma_0\right) \int_0^R \int_{-1}^1 \frac{(1-t^2)dt\rho^{1+2\sigma}d\rho}{p^2/2 - |p|\rho t + \rho^2/2 + \rho + \gamma_0 - z}, \tag{7}$$

which is crucial for finding the eigenvalue of the reduced operator $H(p)$. In Lemma 3 of Section 4, we will derive the following properties of the function $F(p,z)$ in the region $\Omega^- = \{(|p|,z) \mid z \leq z_0(|p|), z \in \mathbb{R}\}$:

- $F(p,z)$ is real analytic and is a decreasing function on z,
- The equation $F(p,z) = 0$ has a unique solution $z = z^*(p)$ when $|p| \leq 1$ and there exists a constant $p_0 > 1$ such that it has no solution when $|p| > p_0$.

Theorem 1 (Spectrum of $H(p)$) *For any $p \in \mathbb{R}^3$, the spectrum of $H(p)$ consists of the essential spectrum $[z_0(|p|), +\infty)$ and the eigenvalue $z^*(p)$, which is a solution of the equation $F(p,z) = 0$ in the region*

$$\Omega^- = \{(|p|,z) \mid z \leq z_0(|p|), z \in \mathbb{R}\}.$$

Moreover, we have the following bound for $z^*(p)$:

$$0 < z^*(p) \leq \frac{7}{2} + \frac{4\pi e^2 R^{1+2\sigma}}{1+2\sigma} + \gamma_0.$$

The eigenvalue $z^*(p)$, when it exists, is the infimum of the spectrum $H(p)$, which we denote by $E_\sigma(p)$. Knowing the exact value of $E_\sigma(p)$ would allow us to calculate the effective mass m_{eff}, defined through

$$E_\sigma(|p|) - E_\sigma(0) = \frac{p^2}{2m_{\text{eff}}} + O(|p|^3)$$

for small p. When $E_\sigma(|p|)$ is a C^2-function in a neighborhood of $p = 0$, as a direct consequence of the preceding definition, we have

$$\frac{1}{m_{\text{eff}}} = \lim_{p \to 0} \frac{\partial^2 E_\sigma(p)}{\partial |p|^2}. \tag{8}$$

Theorem 2 (Effective mass) *The function $E_\sigma(|p|)$ is a C^2-function in a neighborhood of $p = 0$ and the effective mass for $H(p)$ has the following form*:

$$\frac{1}{m_{\text{eff}}} = \frac{1 - \pi e^2 D_{12}(0, \gamma_0)}{1 + \pi e^2 D_{12}(0, \gamma_0)} \tag{9}$$

where

$$D_{12}(p, z) = \int_0^R \int_{-1}^1 \frac{(1 - t^2)\rho^{1+2\sigma} dt d\rho}{p^2/2 - |p|\rho t + \rho^2/2 + \rho + \gamma_0 - z}.$$

We consider the special case as $\sigma \to 0$, which removes the infrared renormalization.

Corollary 1 *For any values of e and R, we have*

$$\lim_{\sigma \to 0} \frac{1}{m_{\text{eff}}} = \frac{1 - (8/3)\pi e^2 \ln(R/2 + 1)}{1 + (8/3)\pi e^2 \ln(R/2 + 1)}. \tag{10}$$

Expansion of the effective mass in terms of the fine structure constant $\alpha = e^2/4\pi$ was done in [8], [2], assuming the constant e be a small. Note that our result gives the effective mass for arbitrary values of e and R, and it is consistent with the results of [8] and [2] when $e^2 \ln(R/2 + 1) \approx o(1)$. Indeed, from formula (10), we can derive that

$$m_{\text{eff}} \approx 1 + \frac{16}{3}\pi e^2 \ln\left(\frac{R}{2} + 1\right).$$

3. The resolvent of $H(p)$

Before proving the main results, we will calculate the resolvent of the operator $H(p)$ defined by (5), in the space $\mathcal{H} = \mathbb{C} \oplus L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$.

For any $p \in \mathbb{R}^3$ and $f = (f_0(p), f_1(p, k, 1), f_1(p, k, 2))^t \in \mathcal{H}$, the matrix form of $H(p)f$ is

$$\begin{pmatrix} \widetilde{T}(p) & -\frac{e}{\sqrt{2}}p\cdot\langle G(1)| & -\frac{e}{\sqrt{2}}p\cdot\langle G(2)| \\ -\frac{e}{\sqrt{2}}p\cdot|G(1)\rangle & \widetilde{L}(p,k)+\frac{e^2}{2}|G(1)\rangle\cdot\langle G(1)| & \frac{e^2}{2}|G(1)\rangle\cdot\langle G(2)| \\ -\frac{e}{\sqrt{2}}p\cdot|G(2)\rangle & \frac{e^2}{2}|G(2)\rangle\cdot\langle G(1)| & \widetilde{L}(p,k)+\frac{e^2}{2}|G(2)\rangle\cdot\langle G(2)| \end{pmatrix}$$
$$\times\begin{pmatrix} f_0(p) \\ f_1(p,k,1) \\ f_1(p,k,2) \end{pmatrix}, \tag{11}$$

where the annihilation and creation operators in $\mathcal{H}$ are denoted by

$$\langle G(\lambda)|v = \int \overline{G(k,\lambda)}v(k)dk \quad \text{and} \quad |G(\lambda)\rangle v = G(k,\lambda)v(k),$$

respectively, for each polarization direction $\lambda = 1, 2$. Here, we have introduced the notation $G(k,\lambda) = e(k,\lambda)g(k)$, and note that

$$\|G\|^2 = \sum_{\lambda=1}^{2}\int |e(k,\lambda)|^2 g^2(k)dk = \frac{4\pi}{1+\sigma}R^{2+2\sigma}.$$

The elements on the diagonal are

$$\widetilde{T}(p) = \frac{1}{2}p^2 + \frac{e^2}{4}\|G\|^2 \tag{12}$$

$$\widetilde{L}(p,k) = \frac{1}{2}(p-k)^2 + \omega(k) + \frac{e^2}{4}\|G\|^2. \tag{13}$$

To find the resolvent of $H(p)$, we need to solve the equation

$$(H(p)-z)f = u \tag{14}$$

for a given $u = (u_0(p), u_1(p,k,1), u_1(p,k,2))^t$.

For ease of writing, we also use the following notation:

$$T = \widetilde{T}(p) - z,$$

$$L = \widetilde{L}(p,k) - z,$$

$$b_\lambda(p,k) = -\frac{e}{\sqrt{2}} p \cdot G(k,\lambda), \qquad \lambda = 1,2,$$

$$N(p,k,\lambda) = b_\lambda(p,k)p + \frac{Te}{\sqrt{2}} G(k,\lambda), \quad \lambda = 1,2.$$

Lemma 1 *For any $z \in \mathbb{C} \setminus \mathbb{R}$, the solution of Equation* (14) *can be written as*

$$f_0(p) = \frac{1}{T}\left[u_0(p) + p \cdot (S(p,z)U^{-1})\right] \tag{15}$$

$$f_1(p,k,\lambda) = \frac{1}{TL}\left[Tu_1(p,k,\lambda) - b_\lambda(p,k)u_0(p) - N(p,k,\lambda)\cdot(S(p,z)U^{-1})\right], \tag{16}$$

where $\lambda = 1,2$,

$$S(p,z) = \frac{e}{\sqrt{2}} \int \sum_{\lambda=1}^{2} \overline{G(k,\lambda)} \left(\frac{1}{L}u_1(p,k,\lambda) - \frac{1}{TL} b_\lambda(p,k)u_0(p)\right) dk,$$

and the matrix $U := (u_{ij})_{i,j=1}^3$ is given by

$$u_{ij} = \begin{cases} a + bp_i^2, & i = j, \\ bp_ip_j, & i \neq j, \end{cases} \tag{17}$$

with $a = \frac{2+D_1+D_2}{2}$ and $b = \frac{D_1-3D_2}{2p^2} - \frac{D_1-D_2}{T}$. Here,

$$D_1(p,z) = \pi e^2 \int_0^R \int_{-1}^1 \frac{\rho^{1+2\sigma} dt d\rho}{p^2/2 - |p|\rho t + \rho^2/2 + \rho + \gamma_0 - z}, \tag{18}$$

$$D_2(p,z) = \pi e^2 \int_0^R \int_{-1}^1 \frac{t^2\rho^{1+2\sigma} dt d\rho}{p^2/2 - |p|\rho t + \rho^2/2 + \rho + \gamma_0 - z}. \tag{19}$$

Proof. By introducing the notation

$$Q = \sum_{\lambda=1,2} \frac{e}{\sqrt{2}} \int \overline{G(k',\lambda)} f_1(p,k',\lambda) dk', \tag{20}$$

Equation (14) can be written using the matrix form (11) of $H(p)$, as

$$T f_0(p) - p \cdot Q = u_0(p)$$
$$b_\lambda(p,k) f_0(p) + L f_1(p,k,\lambda) + \frac{e}{\sqrt{2}} G(k,\lambda) \cdot Q = u_1(p,k,\lambda), \quad \lambda = 1,2. \quad (21)$$

Upon solving (21), we obtain

$$f_0(p) = \frac{1}{T}\left[u_0(p) + p \cdot Q\right], \tag{22}$$

$$f_1(p,k,\lambda) = \frac{1}{TL}\left[T u_1(p,k,\lambda) - b_\lambda(p,k) u_0(p) - N(p,k,\lambda) \cdot Q\right]. \tag{23}$$

To conclude the proof, it suffices to find Q. We substitute (22) and (23) into (20) to get an equation for Q:

$$Q + \frac{e}{T\sqrt{2}} \sum_{\lambda=1,2} \int \frac{1}{L} \overline{G(k,\lambda)} N(p,k,\lambda) \cdot Q dk = S(p,z). \tag{24}$$

For $k \neq 0$, let $\hat{k} := k/|k|$. Using the identity

$$p - (\hat{k}, p)\hat{k} = (e(k,1) \cdot p) e(k,1) + (e(k,2) \cdot p) e(k,2),$$

Equation (24) can be written as

$$\begin{aligned} Q - &\left(\frac{e^2}{2T} \int \frac{g^2(k)}{L} \left(p - (p \cdot \hat{k})\hat{k}\right) dk\right)(p \cdot Q) \\ &+ \left(\frac{e^2}{2} \int \frac{g^2(k)}{L} \left(Q - (Q \cdot \hat{k})\hat{k}\right) dk\right) = S(p,z). \end{aligned} \tag{25}$$

Next, we show that all the integrals in (25) can be reduced to certain combinations of the integrals D_1 and D_2, which are defined in (18) and (19). It is easy to derive that

$$\begin{aligned} \frac{e^2}{2} \int \frac{g^2(k)}{L} dk &= \pi e^2 \int_0^R \int_{-1}^1 \frac{\rho^{1+2\sigma} dt d\rho}{p^2/2 - |p|\rho t + \rho^2/2 + \rho + \gamma_0 - z} \\ &= D_1(p,z), \end{aligned} \tag{26}$$

and hence, we can rewrite Equation (25) in a simple matrix form as

$$Q\left(E+\frac{1}{T}\left(D_1E-C\right)\left(TE-p^tp\right)\right)=S(p,z). \tag{27}$$

Here, E is a 3×3 unit matrix and

$$C=\left(\frac{e^2}{2}\int\frac{g^2(k)k_ik_j}{Lk^2}dk\right)^3_{i,j=1}. \tag{28}$$

To calculate the elements of the matrix C, we introduce the following spherical coordinate system (ρ,φ,θ) where $0\le\varphi<2\pi$ and $0\le\theta<\pi$. We take the zenith direction to be $\vec{l}_1(p)=\hat{p}$, and the azimuth direction to be an orthogonal vector $\vec{l}_2(p)=(p_2,-p_1,0)/p_+$. Here, $p_+=\sqrt{p_1^2+p_2^2}$ and $\hat{p}=p/|p|$. Then, any vector $k=(k_1,k_2,k_3)$ can be written as

$$k=a_1\vec{l}_1(p)+a_2\vec{l}_2(p)+a_3\vec{l}_3(p),$$

where the third orthogonal vector is $\vec{l}_3(p)=(p_1p_3,p_2p_3,-p_1^2-p_2^2)/(|p|p_+)$ and

$$a_1=(k,\vec{l}_1(p))=\rho\cos\theta,$$

$$a_2=(k,\vec{l}_2(p))=\rho\cos\varphi\sin\theta,$$

$$a_3=(k,\vec{l}_3(p))=\rho\sin\varphi\sin\theta.$$

This gives us

$$k^t=\begin{pmatrix}k_1\\k_2\\k_3\end{pmatrix}=\begin{pmatrix}\frac{p_1}{|p|}\rho\cos\theta+\frac{p_2}{p_+}\rho\cos\varphi\sin\theta+\frac{p_1p_3}{|p|p_+}\rho\sin\varphi\sin\theta\\ \frac{p_2}{|p|}\rho\cos\theta-\frac{p_1}{p_+}\rho\cos\varphi\sin\theta+\frac{p_2p_3}{|p|p_+}\rho\sin\varphi\sin\theta\\ \frac{p_3}{|p|}\rho\cos\theta-\frac{p_+}{|p|}\rho\sin\varphi\sin\theta\end{pmatrix}. \tag{29}$$

Now, computing the elements of C, in the aforementioned basis, we get

$$\frac{e^2}{2}\int \frac{g^2(k)k_ik_j}{L|k|^2}dk = \begin{cases} \dfrac{D_1(p^2-p_i^2)+D_2(3p_i^2-p^2)}{2p^2}, & \text{if } i=j, \\ -\dfrac{p_ip_j(D_1-3D_2)}{2p^2}, & \text{if } i\neq j, \end{cases} \tag{30}$$

where D_1 and D_2 are defined in (18) and (19). As an example, let us compute one of the elements of the matrix C:

$$\begin{aligned} c_{23} = c_{32} &= \frac{e^2}{2}\int \frac{g^2(k)k_2k_3}{Lk^2}dk \\ &= \pi e^2 \int_0^R d\rho \int_0^\pi \frac{\rho^{1+2\sigma}}{L}\left(\frac{p_2p_3}{p^2}\cos^2\theta - \frac{p_2p_3}{2p^2}(1-\cos^2\theta)\right)\sin\theta d\theta \\ &= -\frac{p_2p_3}{2p^2}(D_1-3D_2). \end{aligned}$$

Now invoking the identities (26) and (30) in (27), one can obtain the equation $QU = S(p,z)$, with U defined as in (17). Then, substituting $Q = S(p,z)U^{-1}$ into (22) and (23) finally establishes the proof. □

4. The spectrum of $H(p)$

In this section, we describe the spectrum of $H(p)$ for each $p \in \mathbb{R}^3$. We make the decomposition $H(p) = H_0(p) + W(p)$, where

$$H_0(p) = \begin{pmatrix} \widetilde{T}(p) & 0 & 0 \\ 0 & \widetilde{L}(p,k) & 0 \\ 0 & 0 & \widetilde{L}(p,k) \end{pmatrix} \tag{31}$$

and

$$W(p) = \frac{e}{\sqrt{2}}\begin{pmatrix} 0 & -p\cdot\langle G(1)| & -p\cdot\langle G(2)| \\ -p\cdot|G(1)\rangle & \frac{e}{\sqrt{2}}|G(1)\rangle\cdot\langle G(1)| & \frac{e}{\sqrt{2}}|G(1)\rangle\cdot\langle G(2)| \\ -p\cdot|G(2)\rangle & \frac{e}{\sqrt{2}}|G(2)\rangle\cdot\langle G(1)| & \frac{e}{\sqrt{2}}|G(2)\rangle\cdot\langle G(2)| \end{pmatrix}. \tag{32}$$

Note that the spectrum of $H_0(p)$ consists only of the essential spectrum, which is $[z_0(|p|), +\infty)$. Since $H(p)$ is a finite rank perturbation of $H_0(p)$, by Weyl's theorem, the essential spectrum of the operator $H(p)$ remains the same. From (15) and (16) of Lemma 1, one can then see that the only possible addition to the spectrum in the interval $(-\infty, z_0(|p|))$ could be the zeros of the function $\det U$.

The remainder of this section will be devoted to finding these zeros for each $p \in \mathbb{R}^3$. From (17), we derive that

$$\begin{aligned} K(p,z) &:= \det U = a^3 + a^2 b p^2 \\ &= \frac{(D_1 + D_2 + 2)^2}{4T}(T - (D_1 - D_2)p^2 + (D_1 - D_2)T) \\ &= \frac{(D_1 + D_2 + 2)^2}{4T}\left[\frac{1}{2}p^2 - z + \gamma_0 - (D_1 - D_2)\left(\frac{1}{2}p^2 + z - \gamma_0\right)\right]. \end{aligned} \tag{33}$$

Since $(D_1 + D_2 + 2)^2 > 0$, it is important to know the behavior of

$$D_{12} := \frac{1}{\pi e^2}(D_1 - D_2)$$

when finding the zeros of $K(p,z)$. The next lemma gives an estimate for the function D_{12} on the curve $z_0(|p|)$.

Lemma 2 *On the curve $z = z_0(|p|)$, we have the following estimate for D_{12}:*

$$D_{12}(p, z_0(|p|)) \leq \begin{cases} \dfrac{4}{3(1+2\sigma)} \cdot \dfrac{R^{1+2\sigma}}{1-|p|}, & \text{if } |p| < \dfrac{1}{2}, \\ \dfrac{4}{1+2\sigma} \cdot \dfrac{R^{1+2\sigma}}{|p|}, & \text{if } |p| \geq \dfrac{1}{2}. \end{cases} \tag{34}$$

Proof. From (18) and (19), we infer

$$D_{12}(p,z) = \int_0^R \int_{-1}^1 \frac{(1-t^2)\rho^{1+2\sigma} dt d\rho}{p^2/2 - |p|\rho t + \rho^2/2 + \rho + \gamma_0 - z}.$$

When $|p| < 1/2$, we have $z_0(|p|) = p^2/2 + \gamma_0$ and

$$D_{12}\left(p, \frac{p^2}{2} + \gamma_0\right) \le \int_0^R \int_{-1}^1 \frac{(1-t^2)\rho^{2\sigma} dt d\rho}{1-|p|}$$
$$= \frac{4R^{1+2\sigma}}{3(1-|p|)(1+2\sigma)}.$$

When $1/2 \le |p| \le 1$, for any $t \in [-1,1]$, we have the following estimate:

$$\frac{1-t^2}{-|p|t+\rho/2+1} \le \frac{2(1-t)}{-|p|t+\rho/2+1}$$
$$= \frac{2}{|p|} - \frac{2}{|p|} \cdot \frac{\rho/2+1-|p|}{\rho/2+1-|p|t}$$
$$\le \frac{2}{|p|}.$$

Therefore, we conclude

$$D_{12}\left(p, \frac{p^2}{2} + \gamma_0\right) \le \int_0^R \int_{-1}^1 \frac{2\rho^{2\sigma} dt d\rho}{|p|}$$
$$= \frac{4}{1+2\sigma} \cdot \frac{R^{1+2\sigma}}{|p|}.$$

Finally, when $|p| > 1$, we have $z_0(|p|) = |p| - 1/2 + \gamma_0$ and

$$D_{12}\left(p, |p| - \frac{1}{2} + \gamma_0\right) = \int_0^R \int_{-1}^1 \frac{(1-t^2)dt\rho^{1+2\sigma} d\rho}{(1/2)(\rho-|p|+1)^2 + |p|\rho(1-t)}$$
$$\le \int_0^R \int_{-1}^1 \frac{(1-t^2)dt\rho^{2\sigma} d\rho}{|p|(1-t)}$$
$$\le \frac{4}{1+2\sigma} \cdot \frac{R^{1+2\sigma}}{|p|}.$$

Combining the two preceding estimates, we complete the proof. □

Next, we investigate the real solutions of Equation (33) in the region Ω^-, to locate the eigenvalues (if any) of the operator $H(p)$.

Lemma 3 *Let* $(|p|, z) \in \Omega^-$ *with* z *real. The equation* $F(p,z) = 0$ *has a* *unique solution $z = z^*(p)$ when $|p| \leq 1$ and there exists a constant $p_0 > 1$ such that it has no solution when $|p| > p_0$. This constant p_0 satisfies the estimate*

$$p_0 < 4 + \frac{2\pi e^2 R^{1+2\sigma}}{1+2\sigma}.$$

Moreover, when $0 \leq |p| \leq p_0$, the range of z has the following two-sided bound:

$$0 < z \leq \frac{7}{2} + \frac{2\pi e^2 R^{1+2\sigma}}{1+2\sigma} + \gamma_0. \tag{35}$$

Proof. Solving the equation $K(p,z) = 0$ in the region Ω^- is equivalent to solving the equation

$$F(p,z) = \frac{1}{2}p^2 - z + \gamma_0 - \pi e^2 D_{12}(p,z)\left(\frac{1}{2}p^2 + z - \gamma_0\right) = 0. \tag{36}$$

Note that

$$\begin{aligned} F'_z(p,z) &= -1 - \pi e^2 D_{12}(p,z) \\ &- \pi e^2 \left(\frac{1}{2}p^2 + z - \gamma_0\right) \int_0^R \int_{-1}^1 \frac{(1-t^2)\rho^{1+2\sigma} dt d\rho}{(p^2/2 - |p|\rho t + \rho^2/2 + \rho + \gamma_0 - z)^2} < 0 \end{aligned}$$

and therefore, $F(p,z)$ is a decreasing function with respect to z in the region Ω^-, if $p^2/2 + z - \gamma_0 \geq 0$. It is also easy to check that $F(p,z) > 0$ if $z < \gamma_0 - p^2/2$. For $0 < |p| \leq 1$ and $z = z_0(|p|)$, we have

$$F(p,z) = -\pi e^2 p^2 D_{12}\left(p, \frac{1}{2}p^2 + \gamma_0\right) < 0,$$

and therefore, (36) has a solution at least when $0 < |p| \leq 1$.

The upper bound in (35) follows from (34). Indeed, we have

$$\begin{aligned} F(p, z_0(|p|)) &= \frac{1}{2}(|p|-1)^2 - \frac{\pi e^2}{2}(p^2 + 2|p| - 1) D_{12}\left(p, |p| - \frac{1}{2} + \gamma_0\right) \\ &\geq \frac{1}{2}(|p|-1)^2 - (|p|+2)\frac{2\pi e^2 R^{1+2\sigma}}{1+2\sigma}, \end{aligned}$$

and the latter expression is strictly positive if

$$|p| \geq 4 + \frac{4\pi e^2 R^{1+2\sigma}}{1+2\sigma}.$$

It can be written in terms of z as

$$z = |p| - 1/2 + \gamma_0 \geq \frac{7}{2} + \frac{4\pi e^2 R^{1+2\sigma}}{1+2\sigma} + \gamma_0.$$

We prove the lower bound of (35) by contradiction. Assume that there exists some $z \leq 0$ satisfying (36). Let $\mu = \pi e^2$ and rewrite Equation (36) as

$$2(D_{12}d)\mu^2 - ((p^2+2z)D_{12} - 2d)\mu + p^2 - 2z = 0, \tag{37}$$

where $d = R^{2+2\sigma}/(1+\sigma)$. Since $\mu > 0$, only the positive solutions of the equation are of interest, and therefore, the following system of inequalities should hold:

$$\begin{cases} ((p^2+2z)D_{12} - 2d)^2 - 8D_{12}d(p^2-2z) \geq 0, \\ (p^2+2z)D_{12} - 2d \geq 0, \end{cases} \tag{38}$$

which is equivalent to

$$\begin{cases} z' \geq -1 - s + 2\sqrt{2s}, \\ z' \geq s - 1, \end{cases}$$

when $z' \leq 0$. Here,

$$s = \frac{2d}{p^2 D_{12}(p,z)} \quad \text{and} \quad z' = \frac{2z}{p^2}.$$

From the assumption $z' \leq 0$, it follows that $s \leq 1$, which is equivalent to

$$D_{12}(p,z) \geq \frac{2d}{p^2}.$$

The latter inequality is *not* true for any values of e and R. Indeed, note that

$$\begin{aligned} D_{12}(p,z) - \frac{2d}{p^2} &= \int_0^R \int_{-1}^1 \frac{(1-t^2)dt\rho^{1+2\sigma}d\rho}{p^2/2 - |p|\rho t + \rho^2/2 + \rho + \gamma_0 - z} \\ &\quad - \int_0^R \int_{-1}^1 \frac{dt\rho^{1+2\sigma}d\rho}{p^2/2} \\ &= \int_0^R \int_{-1}^1 \frac{2(-(|p|t-\rho)^2/2 - \rho - \gamma_0 + z)}{p^2(p^2/2 - |p|\rho t + \rho^2/2 + \rho + \gamma_0 - z)} \, dt\rho^{1+2\sigma}d\rho \\ &< 0. \end{aligned}$$

Therefore, the equation $K(p,z) = 0$ does not admit any solution in the region Ω^- when $z \le 0$. □

5. Proof of the main results

Summarizing the results from the previous sections, we now prove our main theorems.

Proof of Theorem 1. Repeating the argument at the beginning of Section 4, we prove that the essential spectrum of $H(p)$ is $[z_0(|p|), +\infty)$ and the zeros of the function $K(p,z)$ are the only possible addition to the spectrum in the interval $(-\infty, z_0(|p|))$.

By Lemma 3, for each $p \in \mathbb{R}^3$, there exists a solution to $K(p,z) = 0$ and the range of these values of z belongs to the interval

$$\left[0, \frac{7}{2} + \frac{2\pi e^2 R^{1+2\sigma}}{1+2\sigma} + \gamma_0\right).$$

These solutions are the eigenvalues of $H(p)$ for each p, with eigenfunctions

$$\psi(p,k) = \begin{pmatrix} \dfrac{p^2}{T(p,z^*(p))} \\ -\dfrac{(p \cdot |G(k,1)\rangle)}{L(p,z^*(p))} \left(\dfrac{p^2}{T(p,z^*(p))} + 1\right) \\ -\dfrac{(p \cdot |G(k,2)\rangle)}{L(p,z^*(p))} \left(\dfrac{p^2}{T(p,z^*(p))} + 1\right) \end{pmatrix}.$$

This establishes Theorem 1. □

Proof of Theorem 2. It is easy to show that $E_\sigma(|p|)$ is a C^2-function. Moreover in Lemma 3, we proved the equation $F(p,z)=0$ has a unique solution $z=z^*(p)$ for small p, which is equal to $E_\sigma(|p|)$. Using that $E_\sigma(0)=\gamma_0$, $E'_\sigma(0)=0$, and formula (8), we get the desired result:

$$\left.\frac{\partial^2 E_\sigma(p)}{\partial|p|^2}\right|_{p=0} = -\left.\frac{F''_{|p|}}{F'_z}\right|_{p=0} = \frac{1-\pi e^2 D_{12}(0,\gamma_0)}{1+\pi e^2 D_{12}(0,\gamma_0)}.$$

This establishes Theorem 2. □

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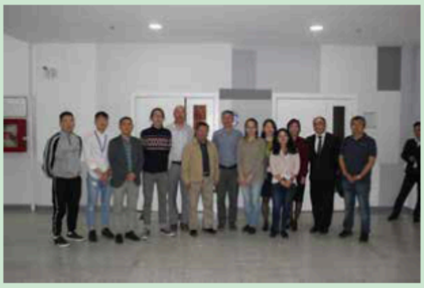
Ulaanbaatar, Mongolia, 2019.



Team members: Z.Uuganbayar, T.Zhanlav, R.Mijiddorj, M.Bayarpurev,, A.Galtbayar, A.Enkhbolor, T.Dultuya, D.Nanzadragchaa and other students

The problem we worked on:

Figure 1: Mobile robot with rotary encoder

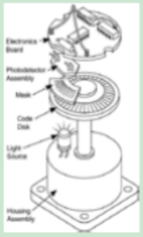


Figure 2: Rotary encoder

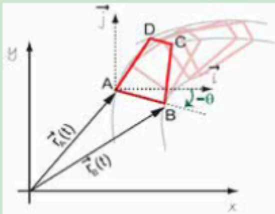


Figure 3: Trajectory of a mobile robot

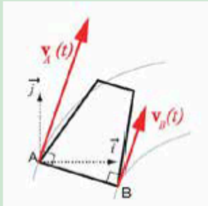


Figure 4: Velocities of the rear wheels

- **Aim:** the real time position determining of the mobile robot equipped with optical rotary encoder.
- **Optical rotary encoder:** An electro-mechanical device that converts the angular position/ motion of shaft to analog/ digital output signals.
- **Method:** Firstly, we should reconstruct velocity of the rear wheels, then by integrating them we calculate the yaw angle and the position of the robot.

Kinematics of mobile robot:

- A Cartesian coordinate systems are chosen and position of rear wheels are given by radius vectors $\vec{r_A}(t)$ and $\vec{r_B}(t)$ (see Fig. 3).
- The corresponding velocity vectors are
$$\vec{V_A}(t) = \frac{d}{dt}\vec{r_A}(t), \vec{V_B}(t) = \frac{d}{dt}\vec{r_B}(t).$$
- Hence, if we know the velocity, we can obtain the position vector by
$$\vec{r_A}(t) = \vec{r_A}(t_0) + \int_{t_0}^{t} \vec{V_A}(\tau)d\tau$$
$$= \vec{r_A}(t_0) + \int_{t_0}^{t} v_A(\tau)\begin{bmatrix}-\sin\theta \\ \cos\theta\end{bmatrix} d\tau$$

Problem formulation (Velocity reconstruction):

The domain of an unknown velocity function *v(t)* is divided into intervals t_i such that the area under the graph of the function on intervals is equal to Δ:

$$\int_{t_{i-1}}^{t_i} v(t) = \Delta$$

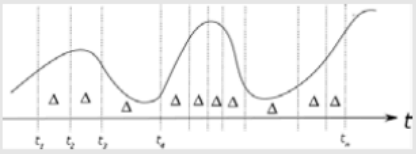
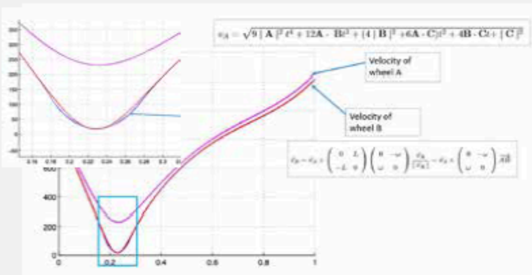


Figure 5: The velocity function *v(t)*

Integro-spline reconstruction of the velocity function:

- We search velocity function as:
$$\hat{v}(t) = \sum_{i=-1}^{n+1} g_i B_i(t),$$
here $B_i(t)$ are third order polynomials.
- We should note that the mesh is **non-uniform.**

Results:



We have successfully reconstructed the velocity function via integro-cubic spline functions on non-uniform mesh and some numerical experiments show that the method has high potential to be applied for real-time position determination of mobile robots.

The Study group workshop on "Collaboration of Industry with Mathematics" will be happy to collaborate with researchers, such as problem proposals or collaboration on problems that Mongolian industry encounters etc.

Presenter: Khongorzul Dorjgotov
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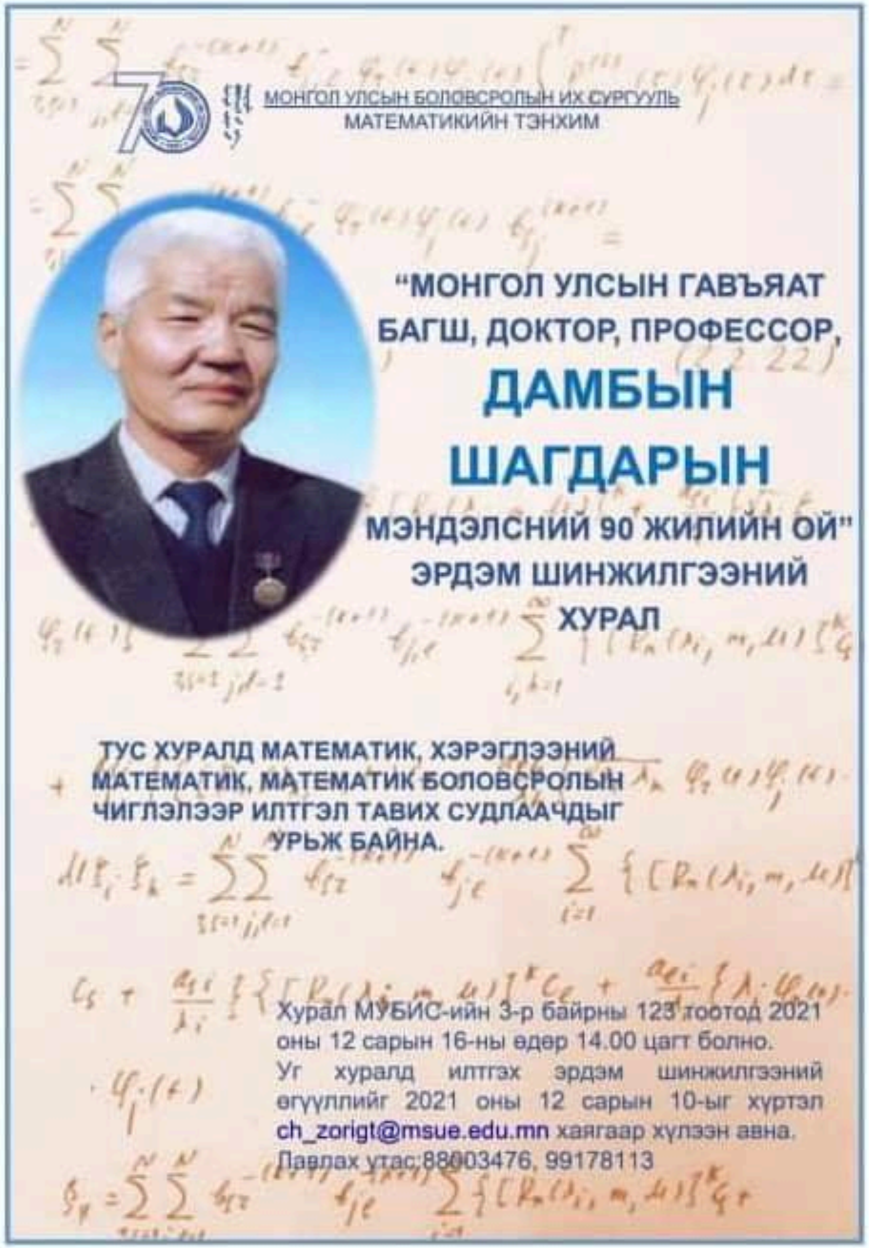
Ulaanbaatar, Mongolia
Email: khongorzul@seas.num.edu.mn

NATIONAL UNIVERSITY OF MONGOLIA 1942

ORGANIZATION FOR WOMEN IN SCIENCE FOR THE DEVELOPING WORLD

6TH GENERAL ASSEMBLY



“МОНГОЛ УЛСЫН ГАВЪЯАТ БАГШ, ДОКТОР, ПРОФЕССОР ДАМБЫН ШАГДАРЫН МЭНДЭЛСНИЙ 90 ЖИЛИЙН ОЙ” ЭРДЭМ ШИНЖИЛГЭЭНИЙ ХУРЛЫН ХӨТӨЛБӨР

2021.12.16

14:00-14:10	**БҮРТГЭЛ (МУБИС 3-р байр, 123 тоот)**
14:10-14:20	**Нээлт**
	Б.Сандагдорж Математикийн тэнхим, МУБИС.
14:20-14:35	**Д.Шагдарын боловсрол, эрдмийн ажлын түүхийн шинжилгээ** **М.Иттэл(МУИС), Т.Ганбаатар(МУБИС), Ж.Дэнсмаа(БХ)**
14:35-14:50	**Экстремаль графийн онолын зарим нээлттэй асуудал** **Б.Хоролдагва(МУБИС)**
14:50-15:05	**Attraction to and repulsion from a subset of the unit sphere for isotropic stable Lévy processes** **Andreas E. Kyprianou, Sandra Palau, Tsogzolmaa Saizmaa**
15:05-15:20	**Суурь-бүтцийн бодлого (Support problem)** **Б.Баясгалан, Б.Баяржаргал(Олонлог төв сургууль)**
15:20-15:35	**Өгөгдлийн эрдэмтэн болоход минь математик надад хэрэг болсон нь. Математик ба Өгөгдлийн шинжлэх ухаан** **Э.Цацрал(Data scientist, Facebook)**
15:35-16:05	**Цайны завсарлага**
16:05-16:20	**Refinement of the upper bound of the radius of a circular integral points set** **Б.Ганбилэг, У.Батзориг(МУИС)**
16:20-16:35	**Хувьсагч коэффициенттэй шугаман диффузи тэгшитгэлийн Ли бүлгээр инвариант байх шийдүүдийн талаар** **З.Ууганбаяр, Д.Хонгорзул(МУИС)**
16:35-16:50	**Мальфаттийн бодлогын онол ба хэрэглээ** **Р. Энхбат(ШУА-ийн Математик, Тоон Технологийн Хүрээлэн)**
16:50-17:05	**Уран сэтгэмжийн бодлого-Шамбалын түлхүүр** **Б.Балдулмаа(Багшийн Сургууль, МУБИС)**
17:05-17:20	**Дээд боловсролын байгууллагын сургалтын үйл ажиллагааны найдвартай байдлыг шинжлэхэд Марковын аргыг хэрэглэх нь** **В.Чимэдлодой(ШУТИС)**
17:20-17:35	**Дадлагын ажил гүйцэтгэх замаар тэнцэтгэл бишийг графикийн аргаар бодох нь** **Б.Анхбаяр, Х.Дэлгэрцэцэг, Д.Батгэрэл, Д.Баасанжав, Ц.Мөнхцэцэг, Б.Цэнд-Аюуш(Ховд аймаг, 6-р сургууль)**
	ХЭЛЭЛЦҮҮЛЭГ, ДУРСАМЖ

МОНГОЛ УЛСЫН ИХ СУРГУУЛЬ

ХЭРЭГЛЭЭНИЙ ШИНЖЛЭХ УХААН, ИНЖЕНЕРЧЛЭЛИЙН СУРГУУЛЬ

ТАВ ДАХЬ УДААГИЙН
ЭРДЭМ ШИНЖИЛГЭЭНИЙ ХУРАЛ

ХЭРЭГЛЭЭНИЙ МАТЕМАТИК 2021

ИЛТГЭЛИЙН ХУРААНГУЙ

УЛААНБААТАР

2022-01-24

Хурлын хөтөлбөр

Нэгдсэн хуралдаан

МУИС-ийн Номын Сангийн 502 тоот
Хуралдааны дарга: С.Батбилэг

9:00–9:20	Бүртгэл
9:20–9:30	Нээлт МУИС, ХШУИС, ХМТ-ийн эрхлэгч С.Батбилэг
Уригдсан илтгэл	
9:30–10:00	Р.Энхбат (ШУА, МТТХ) Applications of Sphere Packing Theory in Economics and Flotation Process.
10:00–10:30	Т.Жанлав (ШУА, МТТХ) Construction of higher-order Newton-like methods for solving nonlinear systems with dynamics
10:30–11:00	Д.Дагвадорж (“Уур амьсгалын өөрчлөлт-Хөгжил” Академи) Уур амьсгалын өөрчлөлтийн судалгааны сүүлийн үеийн ололт, түүнийг хөгжлийн бодлогод тусгах нь
11:00–11:10	Хуралд оролцогчдын зураг авах
11:10–11:30	Цайны завсарлага

Салбар хуралдаан А

МУИС-ийн Номын Сангийн 502 тоот
Хуралдааны дарга: Б.Барсболд

11:30–11:45	А.Батцэнгэл, Д.Цоодол (МУИС, ХШУИС) Нобелийн шагналт нэрт математикч, эдийн засагч Л.В. Канторовичийн мэндэлсний 110 насны ойд
11:45–12:00	Р.Энхбат (ШУА, МТТХ), С.Батбилэг (МУИС, ХШУИС) Бержийн тэнцвэрийн глобаль шийдийг олох нэгэн аргын тухай
12:00–12:15	М.Мэнд-Амар (МУИС, ХШУИС), Б.Дөлгөөн (НТҮГ), Ж. Ганчимэг (ШУТИС) Нийтийн тээврийн хөдөлгөөний урсгалын агент суурьтай симуляцийн загварын тухай
12:15–12:30	А.Галтбаяр(МУИС, ХШУИС), Э.Ганхөлөг (ШУА, МТТХ), М.Мэнд-Амар (МУИС, ХШУИС) Задгай орчин дахь зэсийн баяжмалын исэлдэлтийн математик загвар
12:30–12:45	Ж.Даваажаргал, Ш.Идэрбаяр (ШУА, МТТХ) On the minimization problem of the sum of ratio
12:45–13:00	Э.Энхцолмон (ШУА, МТТХ), Р.Энхбат (ШУА, МТТХ), Д.Цэдэнбаяр (ШУТИС) Extremal Properties of the Cesaro Operator

13:00–14:00	Үдийн цай

Хуралдааны дарга: А.Галтбаяр

14:00–14:15	L.Khadkhuu (МУИС, ШУС), D.Tsedenbayar (ШУТИС) On the accretive properties of powers of the Volterra operator

14:15–14:30	Д.Хонгорзул, З.Ууганбаяр, Э.Алтаннавч (МУИС) Хугацаагаар бутархай эрэмбийн уламжлалтай, хувьсах коеффициенттэй, шугаман диффуз-конвекцийн тэгшитгэлийн зарим инвариант шийдүүд
14:30–14:45	Д.Адъяаням, Э.Азжаргал, Л.Буянтогтох (МУБИС) Bond incident degree indices of stepwise irregular graphs
14:45–15:00	З.Ууганбаяр (МУИС, ШУС), Д.Хонгорзул (МУИС, ХШУИС) Д.Түмэнбаяр (ХААИС) On solutions on Linear Time Fractional Telegraph Equations
15:00–15:15	Г.Баттулга (МУИС, ХШУИС) Options Pricing under Bayesian MS-VAR Process
15:15–15:30	Т.Цэрэннадмид, Т.Дөлтуяа, А.Энхболор (МУИС, ХШУИС) Real-time motion estimation in frequency domain

15:30–15:50	Цайны завсарлага

Хуралдааны дарга: Г.Баттөр

15:50–16:05	Г.Баттулга (МУИС, ХШУИС) Шаардаж буй өгөөжийн үнэлгээ
16:05–16:20	Khulan Myagmar (МУИС), Batkhuyag Ganbaatar (СЭЗИС), Evan J.Douglas (Queensland University of Technology) Using a complex measure of Product Innovativeness to explain Abnormal Financial Returns
16:20–16:35	Ч.Лхагвадулам (ХААИС, ЭЗБС), Д.Цэвээннамжил (ХААИС, ЭЗБС), Б.Барсболд (МУИС, ХШУИС) Солоугийн загварын параметрийн сайжруулалт
16:35–16:50	Т.Цэрэннадмид, Т.Дөлтуяа, А.Энхболор (МУИС, ХШУИС) Noisy translation, rotation and scale stabilization of image sequences

16:50–17:05	Ж.Сонинбаяр (МУИС, ХШУИС) Relationship between ergodicity and mixing in the ergodic theory of dynamical systems
17:05–17:20	Г.Зоригт, Ч.Алдармаа, Л.Хэнмэдэх (ШУТИС, ФТ) Дискрет хувьсагчийн аргаар долгион функцыг тооцоолох

18:30	Оройн зоог

Салбар хуралдаан Б

МУИС-ийн Номын Сангийн 203 тоот
Хуралдааны дарга: Д.Дагвадорж

11:30–11:45	Б.Мөнхбат (УЦУОСМХ) Статистик загвар ашиглан монгол орны агаарын температур, хур тунадасыг өндөр нарийвчлалтай торын зангилааны цэг (грид)-ээр тооцоолох нь (1991-2020)
11:45–12:00	Б.Ганболд, Ш.Отгонсүрэн (МУИС, ХШУИС) Аянга цахилгаантай аадар борооны тохиолдлыг WRF тоон загвараар судалсан дүн
12:00–12:15	У.Пүрэвдорж (УЦУОСМХ), Б.Мөнхбат (УЦУОСМХ), Д.Сандэлгэр (МУИС), С.Эрдэнэсүх (МУИС) Монгол орны био уур амьсгалын нөхцөлийн судалгаа
12:15–12:30	Б.Хүслэн, Ш.Отгонсүрэн (МУИС, ХШУИС) Орог нуур-Түйн голын сав газрын ус зүй, уур амьсгалын судалгаа
12:30–12:45	Д. Номиндарь (ШУТИС, ХШУС) Өгий нуурын сав газрын антропоген стрессийн ерөнхий хэмжүүрийг тодорхойлох нь
12:45–13:00	Makhgal Ganbold (МУИС), Narantuya Damdinsuren, Soninkhishig Nergui, Detection of Bacterial Abundance and Diversity Changes along the Kharaa River

13:00–14:00	Үдийн цай

Хуралдааны дарга: З.Мөнхцэцэг

14:00–14:15	Т.Нарангэрэл (Өвөрхангай УЦУОШТ), Ш.Отгонсүрэн (МУИС), З.Мөнхцэцэг (МУИС), Б.Ганболд (МУИС) Таац голын урсацын горим, түүний өөрчлөлт

14:15–14:30	Гантулга, Мөнхбат (УЦУОСМХ) Уур амьсгалын өөрчлөлт үзэгдлийн төрөл зүйлд нөлөөлөх нь
14:30–14:45	Н.Буянбилиг (МГХАМТИС), Г.Гантуяа (МГХАМТИС), Д.Оюунчимэг (УЦУОСМХ) Монгол орны хүлэмжийн хийн агууламжийн харьцуулсан судалгааны дүнгээс
14:45–15:00	Ц.Бат-Эрдэнэ (Hybrid house XXK), А.Батцэнгэл (МУИС, ХШУИС) Агаарын бохирдлыг бууруулах арга замууд ба сэргээгдэх эрчим хүч
15:00–15:15	Ц.Дамаажав (ХААИС) Монгол сарлагийн хөөврөөр ээрмэл ээрэх явцын математик загвар

15:15–15:35	Цайны завсарлага

Хуралдааны дарга: Г.Баттулга

15:35–15:50	О.Хулан, Ж.Энхбаяр (МУИС, ХШУИС) Монгол дахь авилгалын гэмт хэргийн талаар таамаглахад Машин сургалтыг ашиглах нь
15:50–16:05	М.Цэдэвсүрэн (ШЕЗШСМСТ), Д.Цэвээннамжил (ХААИС, ЭЗБС), Я.Эрдэнэсүрэн (ХААИС, ЭЗБС), Б.Барсболд (МУИС, ХШУИС) Hotelling-ийн T^2 шинжүүрийн хэрэглээ (Шүүхийн тайлан мэдээллийн "хянан шийдвэрлэсэн хэрэг"-ийн жишээн дээр)
16:05–16:20	Makhgal Ganbold (МУИС, ХШУИС) NSO1212: National Statistical Office of Mongolia's Open Data API Handler for R
16:20–16:35	Л.Амарсанаа (ХААИС, ИТС) Програм хангамжийн инженерчлэл дэх математик аргууд
16:35–16:50	Ariunjargal Bat-Erdene (МУИС, ХШУИС) Analysis of Election Data using Potts Model

16:50–17:05	Н.Тогтохбаяр (ХААИС, МААБТС), Д.Баянжаргал (МУИС, ХШУИС) Хурганы мах үйлдвэрлэлийг нэмэгдүүлж, өвөлжих малын тоог цөөлж бэлчээрийн талхагдлыг бууруулах замаар уур амьсгалын өөрчлөлтөд ухаалгаар дасан зохицох арга хэмжээний өртөг өгөөжийн шинжилгээ

18:30	Оройн зоог

“Математик 2022” эрдэм шинжилгээний улсын хэмжээний хурал

Улаанбаатар 2022 он 4 сарын 30

“Математик 2022”

эрдэм шинжилгээний хурал

2022.04.30, Улаанбаатар

Хурлын хөтөлбөр

Цаг	Илтгэл	Хурал удирдагч
10:00-10:05	**Нээлт**	
10:05-10:25	*^{q}N-ийн хуваалт* Илтгэгч: **Б.Баяржаргал**, Т.Хулан (МУИС)	
10:25-10:45	*Finite-dimensional model spaces invariant under composition operators* Илтгэгч: **У.Батзориг**, Г.Батзаяа (МУИС)	
10:45-11:05	*Киппенганы теоремоор Эллиптик мужийн теоремыг батлах нь* Илтгэгч: **Р.Лхагвасүрэн** (МУИС магистрант), У.Батзориг (МУИС)	Б.Баяржаргал
11:05-11:25	*Hotelling-ийн T^2 шинжүүрийг оновчлолын бодлогын заагласан нөхцөлд хэрэглэсэн жишээ* Илтгэгч: **Д.Цэвээннамжил** (ХААИС), Б.Барсболд (МУИС), Д.Мөнхчимэг (ХААИС докторант)	
11:25-11:45	*Conversation Laws for Diffusion Equation* Илтгэгч: **З.Ууганбаяр**, Г.Батзаяа (МУИС)	
11:45-12:05	*Well-posedness of the Fisher’s problem* Илтгэгч: **Ц.Балжинням** (ШУТИС), Ц.Гантулга (МУИС)	
12:05-12:40	**Завсарлага /цай, кофе/**	
12:40-13:00	*On the representation of primes by quadratic norm forms* Илтгэгч: **Г.Баярмагнай**, Д.Пүрэвсүрэн (МУИС)	
13:00-13:20	*On solutions of the Fractional Differential Equations* Илтгэгч: **Б.Ганбилэг** (Олонлог академи), Д.Хонгорзул (МУИС)	
13:20-13:40	*Максимал иррегуляр графын иррегуляр чанар* Илтгэгч: **Ш.Доржсэмбэ**, Б.Хоролдагва, Л.Буянтогтох (МУБИС)	
13:40-14:00	*Ихэр анхны тооны таамаглал* Илтгэгч: **Г.Батзаяа**, Г.Баярмагнай (МУИС)	
14:00-14:20	*Numerical Study of the Boussinesq Equation via GIR method* Илтгэгч: **Ц.Гантулга**, Д.Хонгорзул (МУИС)	
14:20-14:40	*Дифференциал геометр дэх Гилбертийн теоремын тухай* Илтгэгч: **Д.Пүрэвсүрэн** (МУИС)	З.Ууганбаяр
14:40-15:00	*Invariant solutions of the Convection-Diffusion Equation* Илтгэгч: **Д.Хонгорзул**, З.Ууганбаяр (МУИС)	
15:00-15:20	*Херз-Моррийн жинтэй огторгуйд Гильберт төрлийн интеграл оператор заагладах нөхцөл* Илтгэгч: **А.Амарбаяр** (МУИС магистрант), Ц.Батболд (МУИС)	
15:20-15:40	*Хугацааны хувьд бутархай эрэмбийн хувьсах коэффициенттэй конвекц-диффузийн тэгшитгэлийн Ли бүлгийн ангилал* Илтгэгч: **А.Содбаатар** (МУИС докторант), З.Ууганбаяр (МУИС)	
15:40-16:00	*Integral sections of elliptic surfaces and degenerated (2, 3) torus decompositions of a 3-cuspidal quartic* Илтгэгч: **Т.Хулан**, Б.Баяржаргал (МУИС)	
Хаалт		

Хурал зохион байгуулах комисс

“Математик 2022” эрдэм шинжилгээний улсын хэмжээний хурал

Илтгэсэн судалгаа-эрдэм шинжилгээний ажил: 16

Оролцсон байгууллага: 5 (МУИС, ШУТИС, МУБИС, ХААИС, Олонлог Академи)

${}^{q}N$-ийн хуваалт

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Энэхүү илтгэлдээ биноидын Гильберт-Кунзийн функцийг тооцоолох нэгэн аргын талаар авч үзнэ.

Биноид ба зарим тодорхойлолт

N моноид ба хэрэв аливаа $n \in N$ хувьд$\infty + n = n + \infty = \infty$ бол $\infty \in N$-ийг *шингээгч* элемент гэнэ. $(N, +, 0, \infty)$ гэсэн нэгжтэй, шингээгч элементтэй(∞) моноидыг биноид гэнэ. $N^{\bullet} = N \smallsetminus \{\infty\}$.

N тэг биш биноид ба $|N^{\times}|$ нь төгсгөлөг бол хагас эерэг, $N^{\times} = 0$ бол эерэг биноид гэнэ. Энд $N^{\times}$ нь урвуутай элементүүдийн олонлог.

M ба N нь биноид. $\phi : M \to N$ нь моноид гомоморфизм ба ∞_M-ийг ∞_N-д буулгадаг бол биноид гомоморфизм гэнэ.

Хэрэв $a \in N^{\bullet}$-ийн хувьд $a + b = \infty$ эсвэл $b + a = \infty$ гэдгээс $b = \infty$ гэж гардаг бол түүнийг **интеграл** элемент гэнэ. Бүх интеграл элементүүдийн олонлогийг $\mathrm{int}(N)$ гэж тэмдэглэх ба гүйцээлт болох $N \backslash \mathrm{int}(N)$ олонлогийг $\mathrm{int}^c(N)$-ээр тэмдэглэдэг. N коммутатив биноид ба дурын $a \in N^{\bullet}$ нь интеграл элемент болдог бол N-ийг **интеграл биноид** гэнэ.

S олонлог ба $p \in S$ бол (S, p) хосыг **тэмдэглэгдсэн цэгтэй** олонлог гэнэ.

Тодорхойлолт 0.1 *N биноид байг. $+ : N \times S \longrightarrow S$ бинар үйлдэл нь $(n, s) \longmapsto n + s$ дүрмээр тодорхойлогдсон ба дараах чанаруудыг хангадаг бол $S-$ийг* ***N− олонлог*** *гэнэ. Харин $+$ бинар үйлдлээ* ***N− үйлдэл*** *гэж нэрлэнэ. Үүнд:*
1. $\forall s \in S :\ 0 + s = s$
2. $\forall s \in S :\ \infty + s = p$
3. $\forall n \in N :\ n + p = p$
4. $\forall n, m \in N$ *ба* $s \in S :\ (n + m) + s = n + (m + s)$

Лемм 1 *N, M биноидууд ба $\phi : N \to M$ нь биноидын гомоморфизм бол M-ийг N-олонлог гэж үзэж болно.*

S нь N-олонлог ба $\phi : N \to N$ нь биноидын гомоморфизм байг. Тэгвэл S олонлог

$$n \dot{+} s := \phi(n) + s$$

шинэ үйлдлийнхээ хувьд N-олонлог болох ба ${}^{\phi}S$ гэж тэмдэглэнэ.

Аливаа коммутатив N биноид ба $q \in \mathbb{N}$ тооны хувьд

$$N \xrightarrow{[q]} N, \ n \longmapsto qn$$

гомоморфизм тодорхойлж болно. Иймээс бид ${}^{q}N$гэсэн N дээр $[q]$ гомоморфизмоор үйлчилсэн N-ийн элементүүдээс тогтох олонлогтой болов. Лемм ёсоор ${}^{q}N$-ийг тэмдэглэгдсэн цэгтэй N-олонлог гэж үзэж чадна. Мөн $qN := \{qn \mid n \in N\}$, $b + qN := \{b + qn \mid n \in N\}$, гэж тэмдэглэнэ. Энд $b \in N$ ба $[q]N_+$ нь өргөтгөсөн идеал (өөрөөр хэлбэл $< qn | n \in N_+ >$).

Лемм 2 *Хэрэв N төгсгөлөг төрөгдсөн биноид ба q натурал тоо бол ${}^{q}N$нь төгсгөлөг төрөгдссөн N-олонлог байна.*

Өгүүлбэр 0.1 *N коммутатив биноид ба(S, p) нь N-олонлог байг. ХэрэвI нь N-ийн идеал бол $(N/I) \wedge_N S \cong S/(I + S)$ байна.*

Мөрдлөгөө 0.1 *N коммутатив биноид бол ${}^{q}N \wedge_N N/N_+ \cong N/[q]N_+$ байна.*

Энэ мөрдөлгөө нь бидэнд Гильберт-Күнзийн функц болох $\mathrm{HKF}(N, q) = N/[q]N_+$-г тооцоолох боломж олгоно.

$\dot{+}$ үйлдлээр ${}^{q}N$нь N-олонлог болох ба эндээс ${}^{q}N$дээр $\sim_N$ гэсэн эквивалентын харьцаа өгөгдөнө.

Өгүүлбэр 0.2 *N төгсгөлөг төрөгдсөн, хагас эерэг биноид ба $k = k(q)$ нь ${}^{q}N$-ийн N-олонлог үүсгэгчдийн хамгийн цөөн тоо нь бол $\mathrm{HKF}(N, q) = k \cdot |N^{\times}|$ байна.*

N коммутатив биноид ба $q \geqslant 1$ бүхэл тоо байг. Аливаа $b \in N$ хувьд

$$\begin{aligned} {}^{q}P(b) &:= \{a \in N \mid \exists n \in N^{\bullet}, a + qn \in b + qN\} \subseteq N, \\ {}^{q}N(b) &:= \{a \in N \mid a + b \in qN\} \end{aligned}$$

олонлогуудыг тодорхойлж болно. Тэгвэл ${}^{q}P(b)$ нь ${}^{q}N$-ийн N-дэдолонлог , ${}^{q}N(b)$ нь N-олонлог болно.

Зарим үр дүн

Лемм 3 *Хэрэв N интеграл биноид ба q эерэг бүхэл тоо, $b \in N^{\bullet}$ бол ${}^{q}P(b)$ нь ${}^{q}N$-ийн үл задрах хэсэг байна.*

Мөрдлөгөө 0.2 *Хэрэв* N *интеграл биноид ба* q *эерэг бүхэл тоо бол ямар нэг* $T \subseteq N$ *олонлогийн хувьд*

$$^{q}N = \dot{\bigcup_{b \in T}} \ ^{q}P(b)$$

байна.

Мөрдлөгөө 0.3 N *нь төгсгөлөг төрөгдсөн, интеграл биноид байг. Хэрэв* q *эерэг бүхэл тоо бол* $^{q}N = \dot\bigcup_{i=1}^{s} \ ^{q}P(b_i)$ *байх* $b_1, \ldots, b_s \in N$ *элементүүд олдоно.*

Лемм 4 N *нь төгсгөлөг төрөгдсөн, интеграл биноид байг. Хэрэв* q *эерэг бүхэл тоо ба* $b \in N$ *бол* $^{q}P(b) = {}^{q}N(b')$ *байх* $b' \in N$ *олдоно.*

Мөрдлөгөө 0.4 N *нь төгсгөлөг төрөгдсөн, интеграл биноид байг. Хэрэв* q *эерэг бүхэл тоо бол* $^{q}N = \dot\bigcup_{i=1}^{s} \ ^{q}N(b_i)$ *байх* $b_1, \ldots, b_s \in N$ *олдоно.*

Finite dimensional model spaces invariant under composition operators

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Хураангуй

Let $\mathbb{D}$ denote the open unit disk in $\mathbb{C}$ and let $H^2(\mathbb{D})$ be the Hardy space over $\mathbb{D}$. For φ a holomorphic self-map of $\mathbb{D}$, define the composition operator C_φ on $H^2(\mathbb{D})$ by

$$C_\varphi f = f \circ \varphi \quad (f \in H^2(\mathbb{D})). \tag{1}$$

It is well known that $C_\varphi \in B(H^2(\mathbb{D}))$. Here $B(H^2(\mathbb{D}))$ denote the set of all bounded linear operators on $H^2(\mathbb{D})$.

The theory of composition operators is highly interdisciplinary with its natural connections to complex analysis, linear dynamics, complex geometry, and functional analysis.

Conjecture 1 ([2])**.** *Consider the finite Blaschke product* $\theta = \prod_{i=1}^m b_{\alpha_i}^{n_i}$ *corresponding to* $\alpha_1, \dots, \alpha_n \in \mathbb{D}$*. Characterize holomorphic self maps of* $\mathbb{D}$ *such that*

$$C_\varphi Q_\theta \subseteq Q_\theta.$$

Partial answers to the conjecture.

Теорем 0.1. *Let* φ *be a holomorphic self map of* $\mathbb{D}$*,* $\alpha, \beta \in \mathbb{D} \setminus \{0\}, \alpha \neq \beta$*, and suppose* $\theta(z) = \dfrac{z-\alpha}{1-\overline{\alpha}z}\dfrac{z-\beta}{1-\overline{\beta}z}$*. Then* Q_θ *is invariant under* C_φ *if and only if*

$$\varphi(z) = \begin{cases} z, & \text{if } \alpha \neq -\beta \\ z \text{ or } -z, & \text{if } \alpha = -\beta \end{cases}$$

.

Теорем 0.2. *Let* φ *be a holomorphic self map of* $\mathbb{D}$*,* $\alpha, \beta \in \mathbb{D} \setminus \{0\}, \alpha \neq \beta$*, and suppose* $\theta(z) = z\dfrac{z-\alpha}{1-\overline{\alpha}z}\dfrac{z-\beta}{1-\overline{\beta}z}$.

i) For $\alpha + \beta = 0$*,* Q_θ *is invariant under* C_φ *if and only if*

1. φ *is constant or*
2. $\varphi = z$ *or*
3. $\varphi = -z$

ii) For $\alpha + \beta \neq 0$*,* Q_θ *is invariant under* C_φ *if and only if*

1. φ *is constant or*
2. $\varphi = z$ *or*

Номзүй

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Киппенганы теоремоор Эллиптик мужийн теоремыг батлах нь

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Хураангуй: Эллиптик тоон мужын тухай теоремыг алгебрын муруйн чанар болон Киппенганы теоремыг ашиглан хэрхэн баталсаныг авч үзнэ.

Theorem 1 (Эллиптик мужын теорем) *Хэрэв A нь 2×2 хэмжээтэй комплекс элементYYдтэй матриц бөгөөд түүний хувийн утгууд нь λ_1 ба λ_2 байг. Тэгвэл A-ын тоон муж $W(A)$ нь $\frac{1}{2}\operatorname{tr}(A)$ төвтэй, λ_1, λ_2 фокустай, бага тэнхлэгийн урт нь*

$$\sqrt{\operatorname{tr}\left(A^*A\right) - \left|\lambda_1\right|^2 - \left|\lambda_2\right|^2}$$

байх эллипс хэлбэртэй диск байна.

Theorem 2 (Киппенганы теорем) *Хэрэв A нь $n \times n$ хэмжээтэй, элементүүд нь комплекс тоо байх матриц бол түүний тоон муж нь бодит хэсгийнх нь язгууруудынх нь гүдгэр бүрхүүл болдог n зэргийн алгебрийн муруй κ_P оршин байна. Өөрөөр хэлбэл*

$$W(A) = \operatorname{conv}\left(\operatorname{Re}\left(\gamma_{P(x,y,1)}\right)\right).$$

Цаашилбал, хэрэв $H_1 := \frac{A+A^*}{2}$ ба $H_2 := \frac{A-A^*}{2i}$ бол

$$P^\delta = |H_1 u + H_2 v + I_n w|$$

Hotelling-ийн T^2 шинжүүрийг оновчлолын бодлогын заагламтын нөхцөлд хэрэглэсэн жишээ

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2022 оны 4-р сарын 27

Удиртгал

Харилцан хамааралтай p хувьсагчдыг нэгэн зэрэг шинжлэх олон хувьсагчийн шинжүүр нь

$$T^2 = (\bar{X} - \mu_0)' \left(\frac{S}{n}\right)^{-1} (\bar{X} - \mu_0) = n(\bar{X} - \mu_0)' S^{-1} (\bar{X} - \mu_0) \qquad (0.1)$$

$$\underset{(p\times 1)}{\bar{X}} = \frac{1}{n}\sum_{j=1}^{n} X_j, \quad \underset{(p\times p)}{S} = \frac{1}{n-1}\sum_{j=1}^{n}(X_j - \bar{X})(X_j - \bar{X})', \quad \underset{(p\times 1)}{\mu_)} = \begin{bmatrix} \mu_{10} \\ \mu_{20} \\ \vdots \\ \mu_{p0} \end{bmatrix}$$

бөгөөд **Harold Hotelling-ийн T^2 шинжүүр** гэдэг. Энэ шинжүүр $\frac{(n-1)p}{(n-p)}F_{p,n-p}$ тархалтыг үүсгэдэг.

Hotelling-ийн T^2 шинжүүрээр шалгасан кластерийн бүрэлдэхүүнийг шугаман програмчлалын бодлогын заагламтын нөхцөлтэй шууд холбож болох санааг энэ судалгааны ажлаар турших үзсэн. Туршилт тавьсан өгөгдөл нь Монгол Улсын Шүүхийн Ерөнхий Зөвлөлөөс захиалан хийлгэсэн шүүхийг өөрчлөх, шинээр байгуулах, татан буулгах асуудлыг тодорхойлох зорилгын хүрээнд өгөгдсөн суурь судалгааны ажлын үзүүлэлтүүд байсан.

Hotelling-ийн шинжүүрийг кластерийн шинжилгээний үр дүнтэй холбох болсон шалтгааныг тайлбарлавал $z = \frac{x}{y}$ гэсэн олонлог дээр явуулсан кластерийн шинжилгээний үр дүнг x, y олонлог тус бүрээр нь судалгаанд задалж бүлэглэх шаардлага тулгарсан тул хамгийн зохимжтой зөв арга нь энэ шинжүүрийг ашиглах явдал байсан болно. Үүнд: z-шүүгчийн ачаалал, x-шүүхэд шийдвэрлэсэн хэргийн тоо, y-шүүхэд ажилласан шүүгчийн тоо

Хэрэглээний математикийн ахисан түвшний онол, шинжилгээний арга, математикийн хүчтэй программ хангамжийн тусламжтайгаар судалгааны баг захиалагч талын хүссэн үр дүнг амжилттай хүлээлгэн өгсөн. Бодлогын шийд буюу үр дүнг захиалагч хүлээн зөвшөөрсөн тухай дараах холбоосоор орж танилцаж болно.

- https://www.judcouncil.mn/site/news_full/12055
- https://www.judcouncil.mn/site/news_full/11989

Түлхүүр үг: Дисперс, Кластер, Бүрэлдэхүүн хэсэг, Загвар, 2D, 3D дүрслэл

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Conservation laws for diffusion equation

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During last years differential equations with derivatives of fractional order have gained increasing popularity [1–3]. Such equations and their systems accurately model various nonlinear phenomena in many fields including wave studies [4, 5], diffusion processes [6–10, 18] and fluid mechanics [11–13], etc. Hence, a number of effective techniques have been developed to construct exact solutions of these equations such as G'/G-expansion type methods [4, 5, 14–17], variational iteration methods [8, 19], Adomian decomposition methods [7, 18], Lie symmetry methods [9, 11–13, 20, 21] and exp-function methods [29–31], among others. In particular, Lie symmetry analysis provides a generic and efficient algorithmic approach for finding exact solutions to fractional partial differential equations (FPDEs) [22, 24, 26, 27] and systems of FPDEs [11, 23, 25, 28] in explicit forms. The key idea of the Lie symmetry analysis is regarding the tangent structural equations under one or several parameters Lie groups of point transformations. One can construct exact solutions including similarity solutions or more general group-invariant solutions by corresponding symmetry reductions.

In this work, with the help of the Lie symmetry analysis we investigate a nonlinear telegraph system of time–fractional equations in the following form:

$$\begin{cases} \dfrac{\partial^\alpha u}{\partial t^\alpha} = k_1 v_x, \\ \dfrac{\partial^\alpha v}{\partial t^\alpha} = F(u)u_x + G(u), \end{cases} \tag{1}$$

where α is a positive non-integer parameter describing the order of the fractional time derivative; $G(u)$ and $F(u)$ are given sufficiently differentiable functions. Here $F(u)$ is non-constant and $F_u(u) \neq 0$. The Riemann–Liouville fractional derivative operator is defined for u by

$$\frac{\partial^\alpha u}{\partial t^\alpha} := \begin{cases} \dfrac{\partial^n u(x,t)}{\partial t^n}, & \text{for } \alpha \in \mathbb{N}, \\ \dfrac{1}{\Gamma(n-\alpha)} \dfrac{\partial^n}{\partial t^n} \displaystyle\int_0^t \frac{u(x,s)}{(t-s)^{\alpha-n+1}} ds, & \text{for } \alpha \in (n-1, n),\ n \in \mathbb{N}. \end{cases}$$

Telegraph type equations are applicable in several important fields and their applications can be found, e.g., in wave propagation [41], signal analysis [42–44], nonlinear elasticity [19] and random walk theory [6, 45], etc. Thus in particular, the fractional telegraph equations have been considered in various contexts by using different methods. For instance, in [36], Chen et al. studied the analytical solutions of the time-fractional telegraph equation under Dirichlet, Neumann and Robin boundary conditions, respectively. These solutions are derived in forms of the Mittag-Leffler function by using separation of variables. In [38], Srivastava et al. presented analytical solutions to the time-fractional telegraph equations using the so-called reduced differential transformation method (RDTM), subjected to the appropriate initial condition. The RDTM is based on the series expansion technique and is applied in a direct way without using linearization and/or transformation. Next, the telegraph equations which have parabolic asymptotics sometimes better model the anomalous diffusion processes rather than traditional parabolic type equations. In [32], Cascaval et al. discussed the asymptotic aspects of the time-fractional telegraph equations by using the Riemann-Liouville approach. In [37], Ansari obtained a formal solution of the time-fractional telegraph equation by applying a fractional exponential operator. This integral operator itself is obtained via the Bromwich integral for the inverse Mellin transform. Further, the fractional telegraph equation is linked to certain classes of stochastic processes as a stable Brownian motion. In [33], Orsingher and Beghin obtained the Fourier transform of the fundamental solutions of the time-fractional telegraph equations. They showed that some telegraph processes are governed by fractional diffusion equations with respect to the Brownian time. Finally, the space-time fractional telegraph equations are considered by use of the Adomian decomposition in [34], and by means of the Laplace-Fourier transforms in [35], [39] and [40], respectively. In the latter work, Tawfik et al. derived the analytical solution of the space-time fractional telegraph equation in terms of Fox's H function and used it to model an anomalous diffusion process in the case of solar energetic particles transport.

This talk is organized as follows: In the second section we present the basic definitions of fractional calculus. The third section presents the fractional transmission line and the methodology proposed. Fractional differential equations are examined separately, with temporal and spatial derivative, respectively. Finally, we show the complete solution and numerical simulations by taking simultaneously both derivatives (time-space derivatives). In the fourth section we depict our conclusions.

Appendix A

Here we provide the solutions of the auxiliary problems that are used in section **??** for investigating self–adjointness of the fractional nonlinear telegraph system (1).

Recall that the Caputo fractional derivative is given by

$$ {}_t^C\mathrm{D}_T^\alpha f(x,t) = \frac{1}{\Gamma(n-\alpha)} \int_t^T \frac{1}{(\tau - t)^{1+\alpha-n}} \frac{\partial^n}{\partial \tau^n} f(x,\tau) d\tau, \quad n = [\alpha] + 1 \tag{2} $$

The Caputo fractional derivative of the power function possesses the following properties [1]:

$$ {}_t^C\mathrm{D}_T^\alpha (T-t)^k = 0 \quad \text{for} \quad k = 0, 1, \dots, n-1 \tag{3} $$

and

$$ {}_t^C\mathrm{D}_T^\alpha (T-t)^p = \frac{\Gamma(p+1)}{\Gamma(p+1-\alpha)} (T-t)^{p-\alpha} \quad \text{for} \quad p > n-1 \tag{4} $$

Problem 1a Given an equation ${}_t^C\mathrm{D}_T^\alpha g(t) = 0$. Find $g(t)$.
Solution If $0 < \alpha < 1$ then $n = 1$ and $n = 2$ if $1 < \alpha < 2$. From (3) for $k = 0$ and $k = 1$, it immediately follows that

$$ \begin{aligned} & g(t) = c_0 && \text{if } 0 < \alpha < 1 \\ & g(t) = c_1(T-t) + c_0 && \text{if } 1 < \alpha < 2 \end{aligned} \tag{5} $$

where c_0, c_1 are arbitrary constants.

Problem 1b Given an equation ${}_t^C\mathrm{D}_T^\alpha g(t) = A(T-t)^\beta$. Find $g(t)$.
Solution Analogously to the previous solution, we obtain [Kilbas??]

$$ \begin{aligned} & g(t) = A\frac{\Gamma(\beta+1)}{\Gamma(\beta+1+\alpha)}(T-t)^{\beta+\alpha} + c_0 && \text{if } 0 < \alpha < 1 \text{ and } \beta + \alpha > 0 \\ & g(t) = A\frac{\Gamma(\beta+1)}{\Gamma(\beta+1+\alpha)}(T-t)^{\beta+\alpha} + c_1(T-t) + c_0 && \text{if } 1 < \alpha < 2 \text{ and } \beta + \alpha > 1 \end{aligned} \tag{6} $$

from (4), where c_0, c_1 are arbitrary constants.

Problem 2 Given a coupled system

$$\begin{cases} \mu^{(\alpha)} = A\eta \\ \eta^{(\alpha)} = B\mu \end{cases}$$

where $0 < \alpha < 1$. Find μ and η.
Solution We look for a solution in the power series while separating the powers into even for μ and odd for η respectively, i.e.

$$\mu = \sum_{i=0}^{\infty} a_{2i} z^{\alpha \cdot 2i + k} \quad \text{and} \quad \eta = \sum_{i=0}^{\infty} b_{2i+1} z^{\alpha \cdot (2i+1)+k} \quad \text{for} \quad k = 0, 1, 2, \ldots$$

Substituting μ and η into the first equation of the given system and shifting index i for μ back to 0, we have

$$\begin{aligned} \sum_{i=1}^{\infty} a_{2i} \frac{\Gamma(\alpha \cdot 2i + k + 1)}{\Gamma(\alpha \cdot (2i-1) + k + 1)} z^{\alpha \cdot (2i-1)+k} &= \sum_{i=0}^{\infty} a_{2(i+1)} \frac{\Gamma(\alpha \cdot 2(i+1) + k + 1)}{\Gamma(\alpha \cdot (2i+1) + k + 1)} z^{\alpha \cdot (2i+1)+k} \\ &= A \sum_{i=0}^{\infty} b_{2i+1} z^{\alpha \cdot (2i+1)+k} \end{aligned}$$

Equating the corresponding coefficients in the latter equality gives us

$$a_{2(i+1)} = A \frac{\Gamma(\alpha \cdot (2i+1) + k + 1)}{\Gamma(\alpha \cdot 2(i+1) + k + 1)} b_{2i+1}$$

In similar fashion, we have

$$b_{2i+1} = B \frac{\Gamma(\alpha \cdot 2i + k + 1)}{\Gamma(\alpha \cdot (2i+1) + k + 1)} a_{2i}$$

from the second equation. Now using the recurrence in i and alternating substitutions of a and b, we obtain

$$\begin{aligned} a_{2(i+1)} &= A \frac{\Gamma(\alpha \cdot (2i+1) + k + 1)}{\Gamma(\alpha \cdot 2(i+1) + k + 1)} \left(B \frac{\Gamma(\alpha \cdot 2i + k + 1)}{\Gamma(\alpha \cdot (2i+1) + k + 1)} a_{2i} \right) \\ &= \ldots \\ &= (AB)^{i+1} \frac{\Gamma(k+1)}{\Gamma(\alpha \cdot 2(i+1) + k + 1)} a_0 \end{aligned}$$

or

$$a_{2i} = \frac{\Gamma(k+1)}{\Gamma(\alpha \cdot 2i + k + 1)} (AB)^i a_0$$

for even indices of a. In similar manner, it yields

$$b_{2i+1} = \frac{\Gamma(k+1)}{\Gamma(\alpha \cdot (2i+1) + k + 1)} B (AB)^i a_0$$

for odd indices of b. By setting $a_0 = 1$, the solution is

$$\begin{aligned} \mu(z) &= \sum_{i=0}^{\infty} \frac{\Gamma(k+1)}{\Gamma(\alpha \cdot 2i + k + 1)} (AB)^i z^{\alpha \cdot 2i + k} = \Gamma(k+1) z^k \cdot E_{2\alpha, k+1}(ABz^{2\alpha}) \\ \eta(z) &= \sum_{i=0}^{\infty} \frac{\Gamma(k+1)}{\Gamma(\alpha \cdot (2i+1) + k + 1)} B (AB)^i z^{\alpha \cdot (2i+1)+k} = \Gamma(k+1) B z^{\alpha+k} \cdot E_{2\alpha, \alpha+k+1}(ABz^{2\alpha}) \end{aligned} \tag{7}$$

Likewise, we seek a solution in power series with the odd powers for μ and even powers for η. Thus, the final solution of the given system for $k = 0, 1, \dots, n-1$ is

$$\begin{aligned} \mu(z) &= \sum_{k=0}^{n-1} c_{k,1} z^k \cdot E_{2\alpha,k+1}(ABz^{2\alpha}) + \sum_{k=0}^{n-1} c_{k,2} A z^{\alpha+k} \cdot E_{2\alpha,\alpha+k+1}(ABz^{2\alpha}) \\ \eta(z) &= \sum_{k=0}^{n-1} c_{k,1} B z^{\alpha+k} \cdot E_{2\alpha,\alpha+k+1}(ABz^{2\alpha}) + \sum_{k=0}^{n-1} c_{k,2} z^k \cdot E_{2\alpha,k+1}(ABz^{2\alpha}) \end{aligned} \tag{8}$$

where c's are arbitrary constants.

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Well-posedness of the Fisher's problem

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Abstract: In 2009, Prof. R. Picard showed that a number of initial boundary-value problems of classical mathematical physics is generally represented in the linear operator equation and established its well-posedness and causality in a Hilbert space setting. We say that a problem is well-posed if the problem has a unique solution and the solution continuously depends on given data. The independence of the future behavior of a solution until a certain time indicates the causality of the solution. We shall establish the well-posedness and causality of the solution of the evolutionary problems with a perturbation, which is defined by a quadratic form. As an example, we will consider Fisher's problem.

Keywords–Perturbation, Lipschitz continuity, quadratic form, Evolutionary problems.

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On the representation of primes by quadratic norm forms

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Abstract: In this talk we revisit the classical question of which primes can be represented as the absolute value of the norm of an integer in an Euclidean quadratic field K. Finite abelian groups arising from the additive structure of the ring of integers in K were considered to give a clear answer to the question.

1 Introduction.

Let d be a square-free integer, $d \neq 0, 1$. The ring of integers $\mathcal{O}_K$ of the quadratic field $K = \mathbb{Q}(\sqrt{d})$ consists of all complex numbers of the form $a + \omega b$, where a, b are integers and

$$\omega = \begin{cases} \sqrt{d}, & \text{if } d \equiv 2, 3 \pmod 4 \\ \frac{1+\sqrt{d}}{2}, & \text{if } d \equiv 1 \pmod 4 \end{cases}.$$

Throughout, we assume that the field K is Euclidean; that is, there exists a map $\phi : \mathcal{O}_K \setminus \{0\} \to \mathbb{N}$ such that given any a and non zero b in $\mathcal{O}_K$, there exist q and r in $\mathcal{O}_K$ such that

$$a = bq + r \text{ with either } r = 0 \text{ or } \phi(r) < \phi(b).$$

It is well known that there are exactly five Euclidean imaginary quadratic fields (for $d = -11, -7, -3, -2, -1$) and these fields are all norm-Euclidean, *i.e.*, the norm map $N_{K/\mathbb{Q}}$ serves as the map ϕ (see the book by Hardy and Wright). The situation is much different for real quadratic fields. For example, the fields $\mathbb{Q}(\sqrt{14})$ and $\mathbb{Q}(\sqrt{69})$ are Euclidean but not norm-Euclidean (see Harper and Clark). In contrast to the imaginary quadratic fields, there is a conjecture that there are infinitely many Euclidean real quadratic fields. Note that there are exactly 16 norm-Euclidean real quadratic fields (for $d = 2, 3, 5, 6, 7, 11, 13, 17, 19, 21, 29, 33, 37$, 41, 57, 73).

This talk will address the question of which primes can be represented as the absolute value of the norm of an integer in $\mathcal{O}_K$. This is one of the oldest and most studied problem in number theory (see the book by Cox for details). The most classic result in this area is the so-called Fermat's two-squares theorem:

Theorem. Any prime p of the form $4n + 1$ can be expressed as a sum of two squares.

The aim of this talk is to show that the following relatively powerful result can be derived only from the additive structure of $\mathcal{O}_K$ related to subgroups. The proof of Theorem 1 uses the simplest group-theoretic considerations and makes manifest that Fermat's theorem is a simple consequence of the first isomorphism theorem.

Theorem 1. Any prime p not dividing $N_{K/\mathbb{Q}}(\omega)$ can be expressed as the absolute value of the norm of an integer in $\mathcal{O}_K$ if the congruence $N_{K/\mathbb{Q}}(x + \omega) \equiv 0 \pmod p$ has a solution.

For the Gaussian field $K = \mathbb{Q}(\sqrt{-1})$, the condition that $N_{K/\mathbb{Q}}(x+\omega) \equiv 0 \pmod p$ has a solution is equivalent to stating that p is of the form $4n+1$. Indeed, Gauss's quadratic reciprocity law gives us a criterion for deciding whether the congruence $N_{K/\mathbb{Q}}(x + \omega) \equiv 0 \pmod p$ for given integer d and prime p has a solution (see, for example, Brown or Cox).

ON SOLUTIONS OF THE FRACTIONAL DIFFERENTIAL EQUATIONS

Bat-Ochir Ganbileg (Olonlog Academy)
with Khongorzul Dorjgotov (National University of Mongolia)

Fractional differentiation was first discussed in Leibniz's notes. Since that time, almost three centuries ago, fractional differentiation has been developed mainly as a purely theoretical field of mathematics. However, for the last several decades, the application of fractional differentiation to the mathematical modeling of physical problems has become increasingly common, due to the fact that fractional differentiation provides a useful tool for the description of memory and hereditary properties of various materials and processes [1, 2, 3]. In particular, anomalous diffusion processes in complex systems, from charge transport in amorphous semiconductors to bacterial motion, have been successfully modeled with fractional diffusion wave equations [4]. As in the case of integer-order differential equations, various methods, including Adomian decomposition and integral and differential transforms, have been applied to the problem of solving fractional differential equations [5].

In this talk, we derive exact solutions expressed in terms of well-known special functions for fractional ordinary differential equations (FODEs) of the form

$$\frac{d^\alpha}{dz^\alpha}\varphi(z) = z^k\left(\frac{a_m}{\alpha^m}z^m\frac{d^m}{dz^m}\varphi(z) + \frac{a_{m-1}}{\alpha^{m-1}}z^{m-1}\frac{d^{m-1}}{dz^{m-1}}\varphi(z) + \cdots + \frac{a_1}{\alpha}z\frac{d}{dz}\varphi(z) + a_0\varphi(z)\right), \tag{1.1}$$

where $\alpha \in \mathbb{R}_+$, $a_i \in \mathbb{R}$ $(i = 0, \ldots, m)$ and $a_m \neq 0$. Here, for $\alpha \in \mathbb{R}_+$, fractional differentiation is defined in the Riemann-Liouville manner:

$$\frac{d^\alpha}{dz^\alpha}\varphi(z) := \begin{cases} \frac{d^n}{dz^n}\varphi(z), & \text{for } \alpha = n \in \mathbb{N}, \\ \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dz^n}\int_0^z (z-s)^{n-\alpha-1}\varphi(s)ds, & \text{for } \alpha \in (n-1, n) \text{ with } n \in \mathbb{N}. \end{cases} \tag{1.2}$$

Throughout this work, we consider $n \in \mathbb{N}$ satisfying $0 \leq n-1 < \alpha < n$.

It is interesting to consider the forms taken by (1.1) in the particular case that $m = 2$, because these are the cases most commonly considered in scientific and engineering fields. In these cases, we obtain the FODE

$$\frac{d^\alpha}{dz^\alpha}\varphi(z) = a\varphi(z) + \frac{b}{\alpha}z\frac{d}{dz}\varphi(z) + \frac{c}{\alpha^2}z^2\frac{d^2}{dz^2}\varphi(z), \text{ where } a, b, c \in \mathbb{R}. \tag{1.3}$$

In the case that $\alpha = 1$ and $\varphi(z)$ takes the form $\varphi(z) = z^r e^{-\frac{1}{cz}}\phi(\frac{1}{cz})$, (1.3) reduces to Kummer's equation,

$$z^2 \frac{d^2}{dz^2}\phi(z) + \left(2r - \frac{b}{c} + 2 - z\right)\frac{d}{dz}\phi(z) + \left(\frac{b}{c} - r - 2\right)\phi(z) = 0,$$

where $r = -\frac{b-c+\sqrt{(b-c)^2-4ac}}{2c}$. Further, in the case that $\alpha = 2$ and $\varphi(z)$ takes the form $\varphi(z) = (cz^2-1)^{-\frac{b-2c}{4c}}\phi(\sqrt{c}z)$, (1.3) reduces to the associated Legendre differential equation,

$$(1-z^2)\frac{d^2}{dz^2}\phi(z) - 2z\frac{d}{dz}\phi(z) + \left(l(l+1) - \frac{s^2}{1-z^2}\right)\phi(z) = 0,$$

where $l = \frac{-c+\sqrt{b^2+c^2-4ac-2bc}}{2c}$ and $s = \frac{b}{2c} - 1$. The solutions of the above equations can be expressed in terms of Kummer's function and associated Legendre functions, respectively.

Interestingly, (1.3) can also be obtained from the fractional diffusion-wave equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} = c(x)^2 u_{xx} \tag{1.4}$$

with variable diffusion coefficient $c(x) = A(x+B)^k$ or $c(x) = Ae^{kx}$, where A, B and k are real constants. Specifically, if we transform (1.4) by scaling with the similarity variable $z = (x+B)^{\frac{s}{\alpha}}t$ (where s are suitably chosen real numbers) in the case $c(x) = A(x+B)^k$ and with the similarity variable $z = e^{\frac{k}{\alpha}x}t$ in the case $c(x) = Ae^{kx}$, then we obtain (1.3), where the constants a, b, c, a_1, a_2, b_1 and b_2 are expressed in terms of α, A, B and k. Thus, we can obtain exact invariant solutions to (1.4) by obtaining exact solutions to (1.3), respectively. Symmetry reductions of time fractional diffusion-wave equations and systems with variable diffusion coefficients and exact invariant solutions, which can be obtained using the results of the present paper, appear in works by the present authors [6, 7].

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Максимал иррегуляр графын иррегуляр чанар

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Хураангуй: Хэрэв графын оройн зэргийн дараалалд байгаа ялгаатай элементүүдийн тоо нь хамгийн их зэрэгтэйгээ тэнцүү бол уг графыг максимал иррегуляр(maximally irregular) гэдэг. Графын иррегуляр чанарыг

$$irr(G) = \sum_{(u,v)\in E(G)} |d_G(u) - d_G(v)|$$

гэж Албертсон анх 1997 онд тодорхойлсон. (Энд $d_G(u)$ нь u оройн зэрэг) Бид эрэмбэтэй, хамгийн их оройн зэрэг нь Δ байх максимал иррегуляр графын ангийн хувьд хамгийн их иррегуляр чанартай графыг

$$\Delta \leq 0.6n$$

үед тодорхойлсон үр дүнгээ танилцуулах болно.

Ихэр анхны тооны таамаглал

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Хураангуй: Хоорондох зай нь 2 байх анхны тоонуудыг ихэр анхны тоо гэдэг ба ихэр анхны тоонууд төгсгөлгүй олон байх уу? гэдэг асуудал нь тооны онолын хамгийн эртний алдартай асуудлуудын нэг юм. Энэхүү асуудал нь одоогоор шийдэгдээгүй байгаа боловч хоорондох зай нь 70 саяас ихгүй төгсгөлгүй олон анхны тоо оршино гэсэн гайхалтай үр дүнг Хятадын математикч Zhang 2013 онд баталжээ. Zhang-ийн үр дүнгийн дараа нь Terence Tao тэргүүтэй математикчид нийлэн polymath7 төслийг эхлүүлэн дараа дараагийн чухал үр дүнгүүдийг 2014-2015 онуудад баталсан. Тэдний үр дүн нь энэхүү илтгэлийн хүрээнд дээрх үр дүнгүүдийг товч танилцуулна.

Numerical Study of the Boussinesq Equation using Generalized Integral Representation Method

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Abstract

We present discretization schemes based on Generalized Integral Representation Method (GIRM) for numerical study of the Boussinesq wave. The schemes numerically evaluate the coupled Boussinesq equation for three different solitary wave phenomena, namely, propagation of a single soliton, head-on collision of two solitons and reflection of soliton at a fixed boundary. In each case of the soliton interactions, we utilize different Generalized Fundamental Solutions (GFS) along with piecewise constant approximations for the unknown functions. For the case of soliton reflection, time evolution in GIRM is coupled with the Green's function in order to cope with the complicated boundary conditions that arise from the GIRM derivation. For each case of the soliton interactions above, we conduct numerical experiments and obtain satisfactory approximate solutions. The derivation of the numerical schemes are straightforward and it is easy-to-program.

Keywords: Numerical Solution of Boussinesq Wave, Soliton Interactions, Generalized Integral Representation Method (GIRM), Numerical Schemes based on GIRM
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2000 MSC: 65N99, 65L99

April 28, 2022

Гильбертийн теоремын тухай

Д. Пүрэвсүрэн

МУИС, Математикийн тэнхим

Хураангуй: Дифференциал геометрийн гадаргуун онолын гайхамшигт теоремуудын нэг Алдарт Гильбертийн теорем юм. Энэхүү илтгэлд энэхүү теоремын тухай танилцуулж баталгааг хийх ба баталганий явцыг дэлгэрэнгүй хэлэлцэх юм. Мөн энэхүү теоремыг өргөтгөлийн тухай болон гадаргуун онлын шийдэгдээгүй бодлогуудыг танилцуулах болно. Мөн баталгаанд гарах sin-gordon төрлийн шугаман биш тэгшитгэлийн тухай дурдана.

Invariant solutions of the Convection-Diffusion Equations

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1.

Classical telegraph equations have been introduced by Oliver Heavyside back in 1880s, to describe behavior of an electromagnetic wave in a transmission line. The unknown functions of this partial differential equations (PDE) are the voltage and the current along the transmission line and the coefficients are the transmission line parameters, namely, resistance, inductance and capacitance. The telegraph equations are the utmost important equations in electronics industry, specially in designing of high-frequency electronic circuits.

Recently, time fractional derivative version of classical PDEs has been studied extensively due to their effectiveness in modelling physical phenomena. For instance, anomalous diffusion processes in the complex systems such as charge transport in amorphous semiconductors, bacterial motion and so on, have been successfully modeled with fractional diffusion-wave equations. In this article, we consider the time fractional linear telegraph PDEs of the following form:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = a(x)^2 u_{xx} + b(x) u_x \tag{1}$$

where α is any positive real number, $a(x)$ is a sufficiently differentiable, nonzero function and $b(x)$ is a sufficiently differentiable function. Here, fractional differentiation is defined in the Riemann-Liouville manner:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} := \begin{cases} \frac{\partial^n u}{\partial t^n}, & \text{for } \alpha \in \mathbb{N}, \\ \frac{1}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial t^n} \int_0^t \frac{u(x,s)}{(t-s)^{\alpha-n+1}} ds, & \text{for } \alpha \in (n-1, n), \text{ with } n \in \mathbb{N}. \end{cases} \tag{2}$$

In 1987, Bluman and Kumei [8] gave a complete Lie group classification and some invariant solutions for the classical case $\alpha = 1$ and $b(x) = 0$ in (1). In 2015, Huang and Shen studied Lie symmetries of the time fractional telegraph PDE in (1) for any real $\alpha > 0$ and $b(x) = 0$. More recently in our previous work, we gave Lie symmetries and derived corresponding invariant solutions explicitly for $\alpha > 0$ and $b(x) = 0$ [10].

This talk extends results in [10] by considering (1) with nonzero $b(x)$. We present a complete Lie group classification depending on the relationship between $a(x)$ and $b(x)$, and describe the structure of Lie algebras generated by the infinitesimal symmetries of (1). Then, we derive the corresponding optimal systems and the reduce system of ODEs. We explicitly give solutions to the reduced ODEs in terms of well-known special functions: Mittag-Leffler functions, generalized Wright functions, and Fox-H functions. Consequently, using these solutions, we express the invariant solutions of (1).

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Херз-Морриймn жинтэй огторгуйд Гильберт төрлийн интеграл оператор зааглагдах нөхцөл

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Хураангуй: Функцэн огторгуйн нэгэн сонирхолтой анги болох Херзийн огторгуйг Америкийн математикч Херз 1968 онд Фурье хувиргалтын абсолют нийлэлтийн тухай судалгааны ажилдаа анх танилцуулжээ. Энэ огторгуй дээр Гилбертийн оператор зааглагдах тухай үр дүнг 2009, 2012 онуудад ялгаатай нөхцөлтэй байдлаар Жичан нар тогтоосон. Херзийн огторгуйтай холбогддог өөр нэг сонирхолтой огторгуй нь Морриймn огторгуй юм. Эдгээр хоёр огторгуйн шууд өргөтгөл болдог огторгуйг Херз-Морриймn огторгуй гэдэг. Морриймn огторгуйд Гилбертийн оператор маш сайн судлагдсан байдаг. Иймд Гилбертийн операторын зааглалыг Херз-Морриймn огторгуйд тогтоох асуудал зүй ёсоор тавигдана. Энэхүү илтгэлийн хүрээнд дээрх асуудлын хариулт болох тогтоосон үр дүнгээ танилцуулах болно.

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[0]*Key words and phrases.* integral operator, Hilbert operator, Hardy operator, Hardy-Littlewood-Pólya operator

Хугацааны хувьд бутархай эрэмбийн хувьсах коеффициенттэй конвекц-диффузийн тэгшитгэлийн Ли бүлгийн ангилал

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Хувьсах коэффициенттэй хугацааны хувьд бутархай эрэмбийн тухайн уламжлалт дифференциал тэгшитгэлийн шугаман хувьслын системийн ангийг судлахад Ли симметр анализийг ашигладаг. Энд зөвхөн инфинитесимал симметрүүдийг олж аваад зогсохгүй бүлгийн бүрэн ангилалыг гаргаж авсан бөгөөд инвариант шийдүүдийн ангилалыг тооцсон. Коэффициент функцүүдийн хувьд дифференциал тэгшитгэлийн системийн ангийг гурван ангилалд хуваадаг болохыг бид олж мэдэв. Тохиолдол бүрт инфинитесимал симметрүүдийн Ли алгебруудын нэг хэмжээст оптимал системийг тооцоолж, оптимал системд тохирсон reduced системийг олж авсан. Хэрэглээний хувьд эдгээр оптимал системд тохирох инвариант шийдүүдийг closed form-оор гаргаж авдаг.

1. Конвекци-диффузийн тэгшитгэл нь шингэний чанар эсвэл стохастик өөрчлөлтийн шинж чанар бүхий өргөн хүрээний үзэгдлийг илэрхийлдэг. Бөөмийн хөдөлгөөнийг тодорхойлох Фоккер-Планк, Блэк—Шоулзын загварын дагуу сонголтын үнийг үнэлэх Блэк—Шоулз, шахагдашгүй шингэний Навье—Стокс гэх мэт алдартай тэгшитгэлүүдийг математикийн хувьд конвекц-диффузи тэгшитгэлийн тодорхой тохиолдол гэж үзэж болно. Санах ойн нөлөөгөөр диффузийн төрлийн тэгшитгэл дэх хугацааны улалжлалыг бутархай уламжлал болгон өөрчлөх нь төвөгтэй системийн илүү бодитой загварыг боловсруулахад тусална.

2. Бутархай эрэмбийн дифференциал тэгшитгэлийг олон хэрэглээний шинжлэх ухаанд нийлмэл систем, процессыг тайлбарлах маш сайн хэрэглүүр гэж саяханаас хүлээн зөвшөөрсөн бөгөөд тэдгээр нь ердийн дифференциал тэгшитгэлээс илүү өгөгдсөн физик систем эсвэл процессыг илүү нарийвчлалтай загварчилж чаддаг тул тэдний судалгаа улам бүр түгээмэл болж байна. Жишээлбэл, бутархай дифференциалчлал нь аморф хагас дамжуулагч дахь цэнэг дамжуулалт, бактерийн хөдөлгөөн, амьд эс дэх уургийн тархалт зэрэг нарийн төвөгтэй систем дэх хэвийн бус тархалтын процессыг илүү сайн тодорхойлж чаддаг.

Энэ илтгэлд бид дараах хэлбэрийн хугацааны хувьд бутархай эрэмбийн шугаман конвекц-диффузийн тэгшитгэлийн системийн ангийг авч үзсэн:

$$\begin{cases} \frac{\partial^\alpha u}{\partial t^\alpha} = v_x, \\ \frac{\partial^\alpha v}{\partial t^\alpha} = f(x)u_x + g(x)u, \end{cases} \tag{1}$$

Энд α эерэг бодит тоо, $f(x)$, $g(x)$ хангалттай дифференциалчлагдах функцүүд. Бутархай эрэмбийн уламжлалын хувьд дараахаар тодорхойлогдох Риман—Лиувиллийн уламжлал байна:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} := \begin{cases} \frac{\partial^n u}{\partial t^n}, & \text{for } \alpha \in \mathbb{N}, \\ \frac{1}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial t^n} \int_0^t \frac{u(x,s)}{(t-s)^{\alpha-n+1}} ds, & \text{for } \alpha \in (n-1,n), \text{ with } n \in \mathbb{N}. \end{cases}$$

1987 онд Блуман, Кумэй [6] нар (1) системийн $\alpha = 1$ ба $g(x) = 0$ гэсэн сонгодог тохиолдлын хувьд бүрэн Ли бүлгийн ангилал болон зарим инвариант шийдүүдийг өгсөн. 2015 онд Хуан, Шен [7] нар бодит $\alpha > 0$ ба $g(x) = 0$ тохиолдолд (1) тэгшитгэлийн Ли симметрүүдийг судалсан.

Өмнө нь бид [8]-д $\alpha > 0$ болон $g(x) = 0$ тохиолдолд Ли симметрүүдийг өгч, тодорхой инвариант шийдүүдийг гаргаж авсан. Товчхондоо бид $g(x) = 0$ үед (1) тэгшитгэлээр өгөгдсөн системийн ангийн инфинитесимал симметрүүдээр үүсгэгдсэн Ли алгебруудын оптимал системийн бүх хувиргалтанд харгалзах бүлгийн инвариант шийдүүдийг олсон. Системээс үүсгэгдсэн Ли алгебрийн дагуу системийн ангийг гурван тохиолдол болгон ангилдаг. Эхний тохиолдол нь хувьсагч нь ялгагдах reduced тэгшитгэлийг үүсгэдэг, хоёр дахь тохиолдол нь хувьсагч нь ялгагддагүй тэгшитгэлүүдийг үүсгэдэг ба гурав дахь тохиолдол нь хувьсагч нь ялгагдах болон ялгагдахгүй reduced тэгшитгэлүүдийг хоёуланг нь үүсгэдэг.

Энэ илтгэлд бид тэг биш $g(x)$-тэй (1) системийг авч үзэх замаар [8]-д олж авсан үр дүнг өргөтгөсөн болно. Бид $f(x)$ ба $g(x)$ коэффициент функцүүдийн хоорондын хамаарлыг судалж Ли бүлгийн бүрэн ангиллыг болон (1)-ийн инфинитесимал симметрүүдээр үүсгэгдсэн Ли алгебруудын бүтцийг тодорхойлсон. Үүний үр дүнд бид харгалзах оптимал системүүд болон ердийн дифференциал тэгшитгэлийн reduced системийг гаргаж авсан. Цаашлаад Миттаг-Лефлерийн функц, өргөтгөсөн Райттын функц, Фокс-H функц зэрэг тусгай функцүүдээр reduced ердийн дифференциал тэгшитгэлийн шийдүүдийг тодорхой өгсөн. Эндээс эдгээр шийдүүдийг ашиглан бид (1) системийн бүлгийн инвариант шийдүүдийг илэрхийлсэн болно.

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Integral sections of elliptic surfaces and degenerated $(2,3)$ torus decompositions of a 3-cuspidal quartic

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Abstract. In this note, all varieties are defined over the field of complex numbers $\mathbb{C}$. Let d be an even positive integer and let $p(t,x) \in \mathbb{C}[t,x]$ be a polynomial of the form

$$x^3 + a_1(t)x^2 + a_2(t)x + a_3(t) = 0,$$

where $\deg_t a_i(t) \leq id$. Our aim of this note is to consider when $p(t,x)$ has a decomposition of the form

$$(*) \quad p(t,x) = (x - x_o(t))^3 + (c_0(t)x + c_1(t))^2, \quad x_o(t), c_0(t), c_1(t) \in \mathbb{C}[t].$$

The right hand side of $(*)$ is called a $(2,3)$ torus decomposition of the affine curve given by $p(t,x) = 0$.

We will show that the above plane curve has degenerated $(2,3)$ torus decompositions by using arithmetic properties of elliptic surfaces and show that a 3-cuspidal quartic has infinitely many degenerated $(2,3)$ torus decompositions.

Let E be an elliptic curve defined over the rational function field of one variable $\mathbb{C}(t)$ given by

$$E : y^2 = p(t,x),$$

and we denote the set of $\mathbb{C}(t)$-rational points and the point at infinity O by $E(\mathbb{C}(t))$. It is well-known that $E(\mathbb{C}(t))$ becomes an abelian group, O being the zero element. Now our first statement is as follows:

Proposition 1 *Assume that both of plane curves given by*

$$p(t,x) = 0 \quad \textit{and} \quad s^{3d}p(1/s, x'/s^d) = 0$$

have at worst simple singularities in both of (t,x) and (s,x') planes. Then $p(t,x)$ has a decomposition as in $(*)$ if and only if $E(\mathbb{C}(t))$ has a point P of order 3. The polynomial $x_o(t)$ is given by the x-coordinate of P.

As an application of Proposition 1, we have the following theorem:

Theorem 1 *Let $\mathcal{Q}$ be a quartic with 3 cusps and choose a smooth point z_o on $\mathcal{Q}$. There exists a unique irreducible conic $\mathcal{C}$ as follows:*

(i) $\mathcal{C}$ is tangent to $\mathcal{Q}$ at z_o and passes through three cusps of $\mathcal{Q}$.

(ii) Let $F_{\mathcal{Q}}$, $F_{\mathcal{C}}$, and L_{z_o} be defining equations of $\mathcal{Q}$, $\mathcal{C}$ and the tangent line $\mathcal{L}_{z_o}$ of $\mathcal{Q}$ at z_o, respectively. Then there exists a homogeneous polynomial G of degree 3 such that

$$(**) \quad L_{z_o}^2 F_{\mathcal{Q}} = F_{\mathcal{C}}^3 + G^2.$$

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