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Application of Markowitz Model to Mongolian Government Budget

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Abstract

We apply Markowitz portfolio theory to Mongolian economy in order to define optimal budget structure. We assume that the government revenue is a portfolio consisting of seven major taxes and non-tax revenues. We minimize the variance of the portfolio under fixed return of the government revenue. This optimization problem has been solved by the conditional gradient method on MATLAB. Computational results based on Mongolian economic data are provided.

Keywords

Markowitz Model

1. Introduction

Financial portfolio optimization is widely used in mathematics, statistics, economics and engineering. Fundamental breakthrough in the problem of asset allocation and portfolio optimization is dated to Markowitz's Modern Portfolio Theory [1]. It considers rational investors and models with the problem of minimizing the mean-variance of the portfolio with a fixed value for the expected return on the entire portfolio. The model also assumes a market without any taxes or transaction costs, and where short selling is disallowed but assets are infinitely divisible and can be traded with any non-negative fractions.

There are many works devoted to optimization methods and algorithms for solving the portfolio variance minimization problem. This problem belongs to the convex optimization problem so any stationary point found by an optimization method provides a global solution to the problem. Also, the Markowitz model has been extended in various ways in the literature [2]-[13]. Tobin James's work [9] considers the inclusion of risk-free assets in Markowitz model by the development of the Separation theorem which states that in the presence of a risk-free asset, the optimal risky portfolio can be obtained without any knowledge of the investor's preferences.

Sharpe's Capital Asset Pricing Model (CAPM) [14] takes into account the asset's sensitivity to non-diversifiable risk while it is being added to an already existing well-diversified portfolio. It considers the importance of the covariance structure of the returns, the variance of the portfolio and the market premium. The model assumes that the investors are rational and risk-averse, are broadly diversified across a range of investments, and that they cannot influence the prices of the assets. Assumptions regarding trade or transaction costs, short-selling and trades with non-negative fractions do apply from the traditional Markowitz's framework.

Considering the equity markets in perspective, Fernholzs Stochastic Portfolio Theory [2] discusses a descriptive theory that provides a framework for analyzing portfolio behavior and equity market structure that has both theoretical and practical applications.

Portfolio optimization problems have been studied in [3] [12] [15] [16] and [17]. Formulation of Markowitz's portfolio optimization problem is viewed as a quadratic optimization problem. [10] and [18] provides comprehensive literature to convex and numerical optimization methods to solve such a formulation.

[19] explores a global optimization approach to scenario generation and portfolio optimization looking at them as individual problems. [12] proposes a stochastic programming approach for multi-period portfolio optimization. [5] presents a multi-period scenario generation approach to support portfolio optimization and [20] discusses scenario generation, mathematical models and algorithms for the portfolio optimization problem. [21] explores portfolio selection using hierarchical Bayesian analysis and Markov Chain Monte Carlo (MCMC) methods. [4] discusses the portfolio optimization with an envelope-based multi-objective evolutionary algorithm with a variety of non-convex constraints.

[22] solves the portfolio optimization problem using genetic algorithm. [23] applies genetic algorithms in a multi-stage portfolio optimization system. [24] solves the problem with the same method taking into account transaction costs and minimum transaction lot constraints.

[25] examines constrained Markowitz portfolio selection using ant colony optimization. [26] considers multi-objective particle swarm optimization approach to the portfolio optimization problem. In this paper, for solving the variance minimization problem, we use the conditional gradient method [18] which uses a series of linear programming problems. The paper is organized as follows. In Methodology Section, we introduce briefly Markowitz portfolio theory and show how to apply the theory to Mongolian government budget. In Data Description Section, we use Mongolian economic data and construct matrix tables for the proposed model. In the last section, we implement Markowitz model for Mongolian government budget.

2. Methodology

Assume that a government revenue consists of n revenues

$$A = \sum_{i=1}^{n} A_i ,$$

where A is a total government revenue, and A_i is i-th type of revenue, $i = 1, 2, \cdots, n$.

We can consider A as a portfolio of n assets with weights x_i which means $A_i = x_i A, i = 1, 2, \cdots, n$.

Clearly,

$$\sum_{i=1}^{n} x_i = 1, \quad x_i \geq 0, \quad i = 1, 2, \cdots, n .$$

Let $r_1, r_2, \cdots, r_n$ be rates of the tax revenues returns.

These have expected values

$$E(r_1) = \overline{r}_1, E(r_2) = \overline{r}_2, \cdots, E(r_n) = \overline{r}_n.$$

Then the rate of return of the portfolio is

$$r = \sum_{i=1}^{n} x_i r_i .$$

We denote the variance of the return of i-th tax revenue by σ_i^2, the variance of the return of the portfolio by σ^2, and the covariance of the return of i-th revenue with j-th revenue by σ_{ij}. It is well known that [1] [27]

$$\sigma^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j \sigma_{ij} .$$

To find a minimum-variance portfolio, we fix the mean value at same arbitrary value $\overline{r}$. Then we find the optimal portfolio by solving the following minimization problem [1] [27]:

$$\min \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j \sigma_{ij} \tag{1}$$

subject to

$$\sum_{i=1}^{n} x_i \overline{r}_i = \overline{r} \tag{2}$$

$$\sum_{i=1}^{n} x_i = 1 \tag{3}$$

$$x_i \geq 0, \quad i = 1, 2, \cdots, n \tag{4}$$

Note that problem (1)-(4) is convex from a view point of optimization theory. It can be checked that the matrix of covariance $C_{n \times n} = (\sigma_{ij})$ is positive defined. In order to find a solution to problem (1)-(4), we need to write the Lagrangian as

$$L = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j \sigma_{ij} + \lambda_1 \left(\sum_{i=1}^{n} x_i \overline{r}_i - \overline{r} \right) + \lambda_2 \left(\sum_{i=1}^{n} x_i - 1 \right) + \sum_{i=1}^{n} \mu_i x_i$$

taking into account condition (4).

Then if we apply Karush-Kuhn-Tucker optimality condition to problem (1)-(4), we have

$$\begin{cases} \frac{\partial L}{\partial x_i} = \sum_{i=1}^{n} \sigma_{ij} x_j + \lambda_1 \bar{r}_i + \lambda_2 + \mu_i = 0, \ i = 1, 2, \cdots, n \\ \mu_i x_i = 0, \ i = 1, 2, \cdots, n \\ \lambda_1^2 + \lambda_2^2 + \sum_{i=1}^{n} \mu_i^2 > 0, \ \mu_i \geq 0, \ i = 1, 2, \cdots, n \end{cases} \quad (5)$$

To find an optimal solution, we combine system (5) with (2)-(4). It means that

$$\begin{cases} \sum_{i=1}^{n} \sigma_{ij} x_j + \lambda_1 \bar{r}_i + \lambda_2 + \mu_i = 0, \ i = 1, 2, \cdots, n \\ \sum_{i=1}^{n} x_i \bar{r}_i = \bar{r} \\ \sum_{i=1}^{n} x_i = 1 \\ \mu_i x_i = 0, \ i = 1, 2, \cdots, n \\ \mu_i \geq 0, \ i = 1, 2, \cdots, n \end{cases} \quad (6)$$

This nonlinear system has $(3n+2)$ linear and nonlinear equations with $(2n+2)$ unknowns. So it is better to solve problem (1)-(4) by convex optimization methods and algorithm. For instance, it is convenient to solve problem (1)-(4) by conditional gradient method [27] since at each iteration of the algorithm we solve just a linear programming problem.

3. Data Description

For numerical analysis we use the following Mongolian economic data for period 1991-2018 which shows structure of government revenue consisted of tax and nontax revenues (**Tables 1-3**).

Table 1. Weight of government revenue.

	X_1	X_2	X_3	X_4	X_5	X_6	X_7
Year	Income tax	Social security contributions	Property taxes	Taxes on domestic goods & services	Taxes on foreign trade	Other taxes	Non-tax revenue
1991	0.358	0.099	0.001	0.301	0.041	0.013	0.187
1992	0.427	0.071	0.000	0.243	0.113	0.015	0.131
1993	0.493	0.049	0.000	0.245	0.114	0.011	0.087
1994	0.372	0.073	0.000	0.227	0.088	0.024	0.217
1995	0.336	0.109	0.000	0.194	0.066	0.024	0.270
1996	0.280	0.113	0.000	0.229	0.085	0.035	0.258
1997	0.281	0.095	0.000	0.284	0.040	0.036	0.263
1998	0.173	0.109	0.001	0.321	0.006	0.032	0.358
1999	0.147	0.112	0.001	0.352	0.034	0.034	0.320

Continued

2000	0.207	0.108	0.001	0.347	0.062	0.032	0.244
2001	0.147	0.123	0.004	0.379	0.062	0.033	0.253
2002	0.152	0.114	0.007	0.374	0.052	0.054	0.247
2003	0.176	0.118	0.008	0.343	0.059	0.056	0.240
2004	0.202	0.115	0.008	0.343	0.063	0.087	0.182
2005	0.213	0.114	0.008	0.323	0.068	0.100	0.174
2006	0.351	0.082	0.005	0.259	0.053	0.079	0.171
2007	0.345	0.085	0.004	0.219	0.054	0.091	0.201
2008	0.348	0.106	0.004	0.259	0.065	0.090	0.129
2009	0.261	0.132	0.006	0.255	0.058	0.101	0.187
2010	0.312	0.106	0.004	0.277	0.062	0.099	0.139
2011	0.197	0.112	0.004	0.339	0.080	0.135	0.132
2012	0.179	0.138	0.004	0.337	0.067	0.136	0.139
2013	0.187	0.147	0.007	0.323	0.064	0.125	0.146
2014	0.175	0.146	0.008	0.297	0.057	0.138	0.178
2015	0.196	0.174	0.014	0.275	0.054	0.147	0.139
2016	0.173	0.195	0.017	0.327	0.054	0.097	0.137
2017	0.222	0.182	0.018	0.296	0.070	0.081	0.132
2018	0.226	0.176	0.015	0.321	0.074	0.077	0.111

Source: National Statistical Office, https://www.1212.mn/.

Table 2. Government revenue growth.

Year	Income tax	Social security contributions	Property taxes	Taxes on domestic goods & services	Taxes on foreign trade	Other taxes	Non-tax revenue
1992	1.117	0.277	0.000	0.436	3.917	1.010	0.247
1993	4.194	2.125	0.017	3.531	3.540	2.497	1.986
1994	0.127	1.210	3.918	0.381	0.146	2.173	2.711
1995	0.515	1.512	0.800	0.439	0.269	0.681	1.094
1996	−0.060	0.172	−0.174	0.325	0.454	0.633	0.073
1997	0.373	0.150	0.758	0.700	−0.368	0.398	0.395
1998	−0.338	0.227	2.064	0.215	−0.828	−0.019	0.469
1999	−0.059	0.143	0.246	0.221	4.973	0.178	−0.009
2000	0.898	0.299	−0.036	0.324	1.475	0.259	0.025
2001	−0.129	0.395	4.949	0.338	0.211	0.264	0.271
2002	0.123	0.008	0.951	0.073	−0.090	0.768	0.061
2003	0.347	0.199	0.372	0.065	0.328	0.194	0.128
2004	0.477	0.259	0.249	0.285	0.370	1.017	−0.022
2005	0.239	0.165	0.102	0.109	0.274	0.348	0.120

Continued

2006	1.671	0.171	0.092	0.302	0.265	0.285	0.595
2007	0.360	0.434	0.195	0.167	0.422	0.586	0.628
2008	0.164	0.429	0.114	0.365	0.374	0.140	−0.261
2009	−0.311	0.149	0.213	−0.095	−0.176	0.032	0.336
2010	0.874	0.257	0.238	0.701	0.667	0.541	0.163
2011	−0.145	0.429	0.242	0.658	0.745	0.848	0.287
2012	0.045	0.424	0.279	0.145	−0.030	0.164	0.213
2013	0.273	0.297	1.005	0.169	0.165	0.116	0.279
2014	−0.007	0.050	0.139	−0.029	−0.068	0.167	0.291
2015	0.063	0.132	0.725	−0.120	−0.098	0.012	−0.258
2016	−0.109	0.132	0.210	0.205	0.025	−0.332	−0.004
2017	0.546	0.124	0.253	0.088	0.560	0.001	0.160
2018	0.293	0.227	0.078	0.378	0.332	0.214	0.071

Table 3. Covariance matrix of government revenue.

COVAR (X)	X_1	X_2	X_3	X_4	X_5	X_6	X_7
X_1	0.7692	0.2684	−0.2538	0.5033	0.5707	0.3391	0.2542
X_2	0.2684	0.2266	0.1035	0.2415	0.1775	0.2269	0.2444
X_3	−0.2538	0.1035	1.4036	−0.0608	−0.3865	0.1362	0.3116
X_4	0.5033	0.2415	−0.0608	0.4410	0.4042	0.2934	0.2267
X_5	0.5707	0.1775	−0.3865	0.4042	1.7830	0.3059	0.0925
X_6	0.3391	0.2269	0.1362	0.2934	0.3059	0.3925	0.3184
X_7	0.2542	0.2444	0.3116	0.2267	0.0925	0.3184	0.4095

4. Numerical Results

In this section, we implement the Markowitz model for Mongolian economy. We examine government budget revenue structure which depends on seven types of tax and nontax revenues.

Variable x_i is the weight of i-th tax revenue in the portfolio. The Mongolian government budget consists of the following revenues such as income tax, social security contributions, property taxes, taxes on domestic goods and services, taxes on foreign trade, other taxes and non-tax revenues. Table 4 shows the initial values of variables as well as the optimal solution of problem (1)-(4) found by the conditional gradient method on MATLAB.

Thus, the government should take into account these results in fiscal policy decision making.

Table 4. Solution.

Name	Initial value	Optimal value	Change
Income tax	0.255	0.227	−2.8%
Social security contributions	0.118	0.115	−0.3%
Property taxes	0.005	0.018	1.3%
Taxes on domestic goods & services	0.296	0.194	−10.2%
Taxes on foreign trade	0.063	0.040	−2.3%
Other taxes	0.071	0.147	7.6%
Non-tax revenue	0.192	0.260	6.8%

5. Conclusion

We have tested the Markowitz model on Mongolian economic data in order to define optimal structure of the government revenue which consists of 7 components. Since the variance minimization problem was convex quadratic, for solving the problem we have applied the conditional gradient method coded in MATLAB. The numerical solution was obtained. In the same way, we can consider the problem of maximizing the government return subject to variance constraint. But it will be discussed in the next paper.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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