Filomat 36:3 (2022), 801–812
https://doi.org/10.2298/FIL2203801B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Sharp Bounds for Certain m-linear Integral Operators on p-adic Function Spaces

Tserendorj Batbold[a], Yoshihiro Sawano[b], Ganbaatar Tumendemberel[c]

[a] *Department of Mathematics, National University of Mongolia, Ulaanbaatar 14201, Mongolia*
[b] *Department of Mathematics, Chuo University, 1-13-27, Kasuga, 112-8551, Tokyo, Japan*
[c] *Department of Mathematics, New Mongol Institute of Technology, Ulaanbaatar, Mongolia*

Abstract. In this paper, we establish necessary and sufficient conditions for boundedness of m-linear p-adic integral operators with general homogeneous kernel on p-adic Lebesgue spaces and p-adic Morrey spaces, respectively. In each case, we obtain the corresponding operator norms. Also, we deal with some particular examples and compare them with the previously known from the literature.

1. Introduction and Theorems

The goal of this article is to investigate the boundedness property of the m-linear integral operator T_m^p acting on Lebesgue spaces and Morrey spaces over $\mathbb{Q}_p$. Here and below dx stands for the normalized Haar measure on $\mathbb{Q}_p$ satisfying that the measure of $B_o(0)$ is 1. We write $d\mathbf{y} = dy_1 \cdots dy_m$. See [6] for more about $\mathbb{Q}_p$, for example. The symbol $d\mathbf{z}$ will have a similar meaning. We define the m-linear p-adic integral operator T_m^p.

Definition 1.1. *Let m be a positive integer, $f_1, f_2, \ldots, f_m$ be nonnegative locally integrable functions on $\mathbb{Q}_p$, and let $K : \mathbb{R}_+^{m+1} \to [0, \infty)$ be a function homogeneous of degree $-m$. The m-linear p-adic integral operator T_m^p with the kernel K is defined by*

$$T_m^p(f_1, \ldots, f_m)(x) = \int_{\mathbb{Q}_p^m} K(|x|_p, |y_1|_p, \ldots, |y_m|_p) \prod_{i=1}^{m} f_i(y_i) d\mathbf{y}, \quad x \in \mathbb{Q}_p^*.$$

Let $t \in \mathbb{R}$ and $1 < q < \infty$. Then the weighted Lebesgue space $L^q(\mathbb{Q}_p, |x|_p^t)$ is the set of all measurable functions f for which

$$\|f\|_{L^q(\mathbb{Q}_p, |x|_p^t)} = \left(\int_{\mathbb{Q}_p} |f(x)|^q |x|_p^t dx \right)^{\frac{1}{q}} < \infty.$$

Our conclusion in this paper calculates the operator norm of T_m^p.

2020 *Mathematics Subject Classification*. Primary 42B35; Secondary 11F85, 47A63, 47G10
Keywords. m-linear integral operator, Hilbert operator, Hardy operator, Hardy-Littlewood-Pólya operator, p-adic function space
Received: 03 March 2021; Accepted: 01 August 2021
Communicated by Dragan S. Djordjević
The authors would like to thank the National University of Mongolia for supporting this research (Project No. P2020-3990)
Email addresses: tsbatbold@hotmail.com (Tserendorj Batbold), yoshihiro-sawano@celery.ocn.ne.jp (Yoshihiro Sawano), t99504416@gmail.com (Ganbaatar Tumendemberel)

Theorem 1.2. *Let $\alpha_i \in \mathbb{R}$, $1 < q_i < \infty$, $i = 1, 2, \ldots, m$. Define q and α by $1/q = \sum_{i=1}^{m} 1/q_i$ and $\alpha = \sum_{i=1}^{m} \alpha_i$. Assume $1 \le q < \infty$. Then T_m^p extends to a bounded linear operator from $L^{q_1}(\mathbb{Q}_p, |x|_p^{\alpha_1 q_1/q}) \times \cdots \times L^{q_m}(\mathbb{Q}_p, |x|_p^{\alpha_m q_m/q})$ to $L^q(\mathbb{Q}_p, |x|_p^{\alpha})$ if and only if*

$$C_m = \int_{\mathbb{Q}_p^m} K(1, |y_1|_p, \ldots, |y_m|_p) \cdot \prod_{i=1}^{m} |y_i|_p^{-\frac{1}{q_i} - \frac{\alpha_i}{q}} d\mathbf{y} < \infty. \tag{1}$$

Moreover,

$$\|T_m^p\|_{L^{q_1}(\mathbb{Q}_p, |x|_p^{\alpha_1 q_1/q}) \times \cdots \times L^{q_m}(\mathbb{Q}_p, |x|_p^{\alpha_m q_m/q}) \to L^q(\mathbb{Q}_p, |x|_p^{\alpha})} = C_m. \tag{2}$$

A couple of clarifying remarks may be in order.

Remark 1.3.

1. *It should be noticed here that if $m = 1, \alpha = \frac{q}{r} - 1$ and K is a symmetric function, Theorem 1.2 recaptures a result in [5].*
2. *Theorem 1.2 can be located as a passage of the results in [2].*

Next, we give a sharp estimate of m-linear p-adic integral operator T_m^p on the product of weighted p-adic Morrey spaces. Given a measurable set $E \subset \mathbb{Q}_p$, we write $|E|_{\mathrm{H}} = \int_E dx$. Let $B_\gamma(a) = B(a, p^\gamma)$ for $a \in \mathbb{Q}_p$ and $\gamma \in \mathbb{Z}$. Let $\beta \in \mathbb{R}$, $1 \le q < \infty$ and $\lambda \in (-1/q, 0)$. We define the weighted p-adic Morrey space $L^{q,\lambda}(\mathbb{Q}_p, |x|_p^{\beta})$ to be the set of all measurable functions f for which

$$\|f\|_{L^{q,\lambda}(\mathbb{Q}_p, |x|_p^{\beta})} = \sup_{\gamma \in \mathbb{Z}, a \in \mathbb{Q}_p} \|f\|_{B_\gamma(a); q, \lambda, \beta} < \infty,$$

where

$$\|f\|_{B_\gamma(a); q, \lambda, \beta} = \left(\frac{1}{|B_\gamma(a)|_{\mathrm{H}}^{\beta(1+\lambda q)}} \int_{B_\gamma(a)} \left| f(x) \right|^q |x|_p^{\beta} dx \right)^{\frac{1}{q}}.$$

Theorem 1.4. *Let $\alpha, \beta \in \mathbb{R}$, $1 < q < q_i < \infty$, $1/q = \sum_{i=1}^{m} (1/q_i)$, $\beta = \sum_{i=1}^{m} \beta_i$, $\lambda = \sum_{i=1}^{m} \lambda_i$, $-1/q_i < \lambda_i < 0$ and $f_i \in L^{q_i, \lambda_i}(\mathbb{Q}_p, |x|_p^{\beta_i q_i/q})$, $i = 1, \ldots, m$. If*

$$L_m = \int_{\mathbb{Q}_p^m} K(1, |y_1|_p, \ldots, |y_m|_p) \cdot \prod_{i=1}^{m} |y_i|_p^{\alpha\left(\frac{1}{q_i} + \lambda_i\right) - \frac{1}{q_i} - \frac{\beta_i}{q}} d\mathbf{y} < \infty, \tag{3}$$

then T_m^p is bounded from $L^{q_1,\lambda_1}(\mathbb{Q}_p, |x|_p^{\beta_1 q_1/q}) \times \cdots \times L^{q_m,\lambda_m}(\mathbb{Q}_p, |x|_p^{\beta_m q_m/q})$ to $L^{q,\lambda}(\mathbb{Q}_p, |x|_p^{\beta})$ with the operator norm less than or equal to L_m. Moreover, if $\lambda_1 q_1 = \cdots = \lambda_m q_m$ and $\alpha > 0$, then

$$\|T_m^p\|_{L^{q_1,\lambda_1}(\mathbb{Q}_p, |x|_p^{\beta_1 q_1/q}) \times \cdots \times L^{q_m,\lambda_m}(\mathbb{Q}_p, |x|_p^{\beta_m q_m/q}) \to L^{q,\lambda}(\mathbb{Q}_p, |x|_p^{\beta})} = L_m. \tag{4}$$

It might be interesting to compare Theorem 1.4 with the results in [1]. In fact, in the setting of Morrey spaces, we learn that the maximizer exists.

For the proof of Theorem 1.2 and 1.4, we employ the following two facts. First of all, $|xy|_p = |x|_p |y|_p$ for all $x, y \in \mathbb{Q}_p$. Next, we have

$$|\{|x|_p \le p^\gamma\}|_{\mathrm{H}} = p^\gamma,$$

$$|\{|x|_p = p^\gamma\}|_{\mathrm{H}} = |\{|x|_p \le p^\gamma\}|_{\mathrm{H}} - |\{|x|_p \le p^{\gamma-1}\}|_{\mathrm{H}} = \left(1 - \frac{1}{p}\right)p^\gamma,$$

if $\gamma \in \mathbb{Z}$. A direct consequence is that we have the p-adic change of variables

$$\int_{\mathbb{Q}_p} f(px)dx = p\int_{\mathbb{Q}_p} f(x)dx$$

for any $f \in L^1(\mathbb{Q}_p)$.

As is seen from these facts, the measure of $\mathbb{Q}_p$ satisfies the doubling condition: $|B_{\gamma+1}(a)| = p|B_\gamma(a)|$ for $\gamma \in \mathbb{Z}$. Thus, the weighted Morrey space $L^{q,\lambda}(\mathbb{Q}_p, |x|_p^\beta)$ can be considered as the weighted case of Morrey spaces considered in the paper [3].

2. Proof of Theorem 1.2

We will start with considering an example of $L^{q'}(\mathbb{Q}_p, |x|_p^{\alpha q'/q})$-functions. For $0 < \varepsilon < 1$, $\alpha \in \mathbb{R}$ and $Q > 0$, we set

$$f^\varepsilon(y) = \begin{cases} 0, & |y|_p < 1, \\ |y|_p^{-\frac{1}{q'} - \frac{\alpha}{q} - \frac{Q\varepsilon}{q'}}, & |y|_p \ge 1. \end{cases}$$

We calculate the $L^{q'}(\mathbb{Q}_p, |x|_p^{\alpha q'/q})$-norm of f^ε.

Lemma 2.1. *We have*

$$\|f^\varepsilon\|^{q'}_{L^{q'}(\mathbb{Q}_p, |x|_p^{\alpha q'/q})} = \frac{1 - p^{-1}}{1 - p^{-\varepsilon Q}}.$$

Proof. We calculate

$$\begin{aligned} \|f^\varepsilon\|^{q'}_{L^{q'}(\mathbb{Q}_p, |x|_p^{\alpha q'/q})} &= \int_{|y|_p \ge 1} |y|_p^{-1-Q\varepsilon} dy \\ &= \sum_{i=0}^{\infty} \int_{|y|_p = p^i} |y|_p^{-1-Q\varepsilon} dy \\ &= \sum_{i=0}^{\infty} p^{-i(1+Q\varepsilon)} |\{|y|_p = p^i\}|_{\mathrm{H}} \\ &= \sum_{i=0}^{\infty} p^{-iQ\varepsilon}\left(1 - \frac{1}{p}\right). \end{aligned}$$

Since $p > 1$, the series above converges and we obtain the desired result. □

We prove Theorem 1.2. First we prove the sufficiency of the theorem. Let $f_i \in L^{q_i}(\mathbb{Q}_p, |x|_p^{\alpha_i q_i'} q_i)$ for each i. Suppose that (1) holds. Write the left-hand side out fully and use the triangle inequality to have

$$\begin{aligned} \|T_m^p(f_1, \ldots, f_m)\|_{L^q(\mathbb{Q}_p, |x|_p^\alpha)} &= \left(\int_{\mathbb{Q}_p} \left|\int_{\mathbb{Q}_p^m} K(|x|_p, |y_1|_p, \ldots, |y_m|_p) \cdot \prod_{i=1}^m f_i(y_i) d\mathbf{y}\right|^q |x|_p^\alpha dx\right)^{\frac{1}{q}} \\ &\le \left(\int_{\mathbb{Q}_p} \left(\int_{\mathbb{Q}_p^m} K(|x|_p, |y_1|_p, \ldots, |y_m|_p) \prod_{i=1}^m |f_i(y_i)| d\mathbf{y}\right)^q \cdot |x|_p^\alpha dx\right)^{\frac{1}{q}}. \end{aligned}$$

By the p-adic change of variables and the fact that $|xz|_p = |x|_p|z|_p$ for all $x, z \in \mathbb{Q}_p$, we obtain

$$\|T_m^p(f_1,\dots,f_m)\|_{L^q(\mathbb{Q}_p,|x|_p^\alpha)} \le \left(\int_{\mathbb{Q}_p}\left(\int_{\mathbb{Q}_p^m} K(1,|z_1|_p,\dots,|z_m|_p)\prod_{i=1}^m |f(xz_i)|d\mathbf{z}\right)^q |x|_p^\alpha dx\right)^{\frac{1}{q}}.$$

Using Minkowski's inequality, we have

$$\|T_m^p(f_1,\dots,f_m)\|_{L^q(\mathbb{Q}_p,|x|_p^\alpha)} \le \int_{\mathbb{Q}_p^m}\left(\int_{\mathbb{Q}_p}\prod_{i=1}^m |f(xz_i)|\cdot|x|_p^\alpha dx\right)^{\frac{1}{q}}\cdot K(1,|z_1|_p,\dots,|z_m|_p)d\mathbf{z}.$$

Now, we invoke the assumption $\alpha = \sum_{i=1}^m \alpha_i$ and we obtain

$$\|T_m^p(f_1,\dots,f_m)\|_{L^q(\mathbb{Q}_p,|x|_p^\alpha)} \le \int_{\mathbb{Q}_p^m} K(1,|z_1|_p,\dots,|z_m|_p)\cdot\left(\int_{\mathbb{Q}_p}\prod_{i=1}^m (|f_i(xz_i)|^q\cdot|x|_p^{\alpha_i})dx\right)^{\frac{1}{q}} d\mathbf{z}.$$

Using Hölder's inequality, we have

$$\begin{aligned}\|T_m^p(f_1,\dots,f_m)\|_{L^q(\mathbb{Q}_p,|x|_p^\alpha)} &\le \int_{\mathbb{Q}_p^m} K(1,|z_1|_p,\dots,|z_m|_p)\cdot\left(\prod_{i=1}^m\left(\int_{\mathbb{Q}_p}|f_i(xz_i)|^q\cdot|x|_p^{\alpha_i}\right)^{\frac{q_i}{q}}dx\right)^{\frac{1}{q_i}} d\mathbf{z}\\ &= C_m\cdot\prod_{i=1}^m \|f_i\|_{L^{q_i}(\mathbb{Q}_p,|x|_p^{\alpha_i q_i/q})}.\end{aligned}$$

Thus, we obtain

$$\|T_m^p\|_{L^{q_1}(\mathbb{Q}_p,|x|_p^{\alpha_1 q_1/q})\times\cdots\times L^{q_m}(\mathbb{Q}_p,|x|_p^{\alpha_m q_m/q})\to L^q(\mathbb{Q}_p,|x|_p^\alpha)} \le C_m. \tag{5}$$

Assume instead that T_m^p is bounded on $L^q(\mathbb{Q}_p,|x|_p^\alpha)$. For $0<\varepsilon<1$ and $i=1,2,\dots,m$, we take

$$f_i^\varepsilon(y_i) = \begin{cases} 0, & |y_i|_p < 1,\\ |y_i|_p^{-\frac{1}{q_i}-\frac{\alpha_i}{q}-\frac{\varepsilon}{q_i}}, & |y_i|_p \ge 1.\end{cases}$$

Then $f_i \in L^{q_i}(\mathbb{Q}_p,|x|_p^{\alpha_i q_i/q})$. We estimate

$$\|T_m^p(f_1^\varepsilon,\dots,f_m^\varepsilon)\|_{L^q(\mathbb{Q}_p,|x|_p^\alpha)} = \left(\int_{\mathbb{Q}_p}\left|\int_{\mathbb{Q}_p^m} K(|x|_p,|y_1|_p,\dots,|y_m|_p)\prod_{i=1}^m f_i^\varepsilon(y_i)d\mathbf{y}\right|^q |x|_p^\alpha dx\right)^{\frac{1}{q}}$$

from below. Write $D(a) = \left\{(y_1,y_2,\dots,y_m):\ |y_i|_p \ge \frac{1}{|a|_p},\ i=1,2,\dots,m\right\}$ for $a\in\mathbb{Q}_p$. Then we have

$$\begin{aligned}\|T_m^p(f_1^\varepsilon,\dots,f_m^\varepsilon)\|_{L^q(\mathbb{Q}_p,|x|_p^\alpha)} &\ge \left(\int_{|x|_p\ge1}\left(\int_{D(1)} K(|x|_p,|y_1|_p,\dots,|y_m|_p)\prod_{i=1}^m f_i^\varepsilon(y_i)d\mathbf{y}\right)^q |x|_p^\alpha dx\right)^{\frac{1}{q}}\\ &= \left(\int_{|x|_p\ge1}\left(\int_{D(x)} K(1,|y_1|_p,\dots,|y_m|_p)\cdot\prod_{i=1}^m |xy_i|_p^{-\frac{1}{q_i}-\frac{\alpha_i}{q}-\frac{\varepsilon}{q_i}}d\mathbf{y}\right)^q\cdot|x|_p^\alpha dx\right)^{\frac{1}{q}}.\end{aligned}$$

Now take $\varepsilon = p^{-k}, k \in \mathbb{N}$. We abbreviate

$$C_m(\varepsilon) = \int_{D(1)} K(1, |y_1|_p, \dots, |y_m|_p) \prod_{i=1}^m |y_i|_p^{-\frac{1}{q_i} - \frac{\alpha_i}{q} - \frac{\varepsilon}{q_i}} d\mathbf{y}.$$

We have

$$\begin{aligned}
\|T_m^p(f_1^\varepsilon, \dots, f_m^\varepsilon)\|_{L^q(\mathbb{Q}_p, |x|_p^\alpha)} &\geq \left(\int_{|x|_p \geq |\varepsilon|_p} C_m(\varepsilon)^q |x|_p^{-1-\varepsilon} dx \right)^{\frac{1}{q}} \\
&= C_m(\varepsilon) \left(\int_{|x|_p \geq |\varepsilon|_p} |x|_p^{-1-\varepsilon} dx \right)^{\frac{1}{q}} \\
&= C_m(\varepsilon) \left(\int_{|z|_p \geq 1} |z|_p^{-1-\varepsilon} dz \right)^{\frac{1}{q}} \cdot |\varepsilon|_p^{-\frac{\varepsilon}{q}}.
\end{aligned}$$

Thanks to Lemma 2.1, we have

$$\|T_m^p(f_1^\varepsilon, \dots, f_m^\varepsilon)\|_{L^q(\mathbb{Q}_p, |x|_p^\alpha)} \geq C_m(\varepsilon) \left(\frac{1-p^{-1}}{1-p^{-\varepsilon}} \right)^{\frac{1}{q}} \cdot |\varepsilon|_p^{-\frac{\varepsilon}{q}} = C_m(\varepsilon) \prod_{i=1}^m \|f_i^\varepsilon\|_{L^{q_i}\left(\mathbb{Q}_p, |x|_p^{\frac{\alpha_i q_i}{q}}\right)} \cdot |\varepsilon|_p^{-\frac{\varepsilon}{q}}.$$

Let $k \to \infty$. Then by Fatou's Lemma, we have

$$\|T_m^p\|_{L^{q_1}(\mathbb{Q}_p, |x|_p^{\alpha_1 q_1/q}) \times \cdots \times L^{q_m}(\mathbb{Q}_p, |x|_p^{\alpha_m q_m/q}) \to L^q(\mathbb{Q}_p, |x|_p^\alpha)} \geq C_m. \tag{6}$$

Then by (5) and (6), we get

$$\|T_m^p\|_{L^{q_1}(\mathbb{Q}_p, |x|_p^{\alpha_1 q_1/q}) \times \cdots \times L^{q_m}(\mathbb{Q}_p, |x|_p^{\alpha_m q_m/q}) \to L^q(\mathbb{Q}_p, |x|_p^\alpha)} = C_m.$$

Theorem 1.2 is proved.

3. Proof of Theorem 1.4

Unlike the weighted Lebesgue spaces $L^{q',\lambda}(\mathbb{Q}_p, |x|_p^{\beta q'/q})$ contains a power function.

Lemma 3.1. *Let* $\hat{f}(x) = |x|_p^{\frac{\alpha}{q'} + \alpha\lambda - \frac{1}{q'} - \frac{\beta}{q}}, x \in \mathbb{Q}_p$ *for* $\alpha > 0, \beta \in \mathbb{R}$ *and* $-1/q' < \lambda < 0$. *Then* $\hat{f} \in L^{q',\lambda}(\mathbb{Q}_p, |x|_p^{\beta q'/q})$.

Proof. Consider the following two cases.

(i) If $|a|_p > p^\gamma$ and $x \in B_\gamma(a)$, then $|x|_p = \max\{|x-a|_p, |a|_p\} > p^\gamma$. Since $-\frac{1}{q'} < \lambda < 0$, we have

$$\begin{aligned}
\frac{1}{|B_\gamma(a)|_H^{\alpha(1+\lambda q')}} \int_{B_\gamma(a)} \left|\hat{f}(x)\right|^{q'} |x|_p^{\frac{\beta q'}{q}} dx &= \frac{1}{|B_\gamma(a)|_H^{\alpha(1+\lambda q')}} \int_{B_\gamma(a)} |x|_p^{\alpha + \alpha\lambda q' - 1} dx \\
&< \frac{1}{|B_\gamma(a)|_H^{\alpha(1+\lambda q')}} \int_{B_\gamma(a)} p^{\alpha\gamma + \alpha\gamma\lambda q' - \gamma} dx = 1.
\end{aligned}$$

(ii) If $|a|_p \leq p^\gamma$ and $x \in B_\gamma(a)$, then $|x|_p \leq \max\{|x-a|_p, |a|_p\} \leq p^\gamma$, so $x \in B_\gamma$. In this case $B_\gamma(a) = B_\gamma$, thus

$$\begin{aligned}
\frac{1}{|B_\gamma(a)|_H^{\alpha(1+\lambda q')}} \int_{B_\gamma(a)} \left|\hat{f}(x)\right|^{q'} |x|_p^{\frac{\beta q'}{q}} dx &= \frac{1}{|B_\gamma|_H^{\alpha(1+\lambda q')}} \int_{B_\gamma} |x|_p^{\alpha + \alpha\lambda q' - 1} dx \\
&= \frac{1 - p^{-1}}{1 - p^{-\alpha(1+\lambda q')}}.
\end{aligned}$$

From the above estimates, we get that $\hat{f} \in L^{q',\lambda}(\mathbb{Q}_p, |x|_p^{\beta q'/q})$. $\square$

We prove Theorem 1.4.

Suppose that (3) holds. By the p-adic change of variables, we have

$$T_m^p(f_1, \dots, f_m)(x) = |x|_p^m \int_{\mathbb{Q}_p^m} K(|x|_p, |xy_1|_p, \dots, |xy_m|_p) \prod_{i=1}^m f_i(xy_i) d\mathbf{y}.$$

Since K is homogeneous of degree $-m$, we obtain

$$T_m^p(f_1, \dots, f_m)(x) = \int_{\mathbb{Q}_p^m} K(1, |y_1|_p, \dots, |y_m|_p) \prod_{i=1}^m f_i(xy_i) d\mathbf{y}.$$

Let $\gamma \in \mathbb{Z}$ and denote $yB_\gamma(a) = B(ya, |y|_p p^\gamma)$. By Minkowski's inequality, we have

$$\begin{aligned}
&\left(\frac{1}{|B_\gamma(a)|_{\mathrm{H}}^{\alpha(1+\lambda q)}} \int_{B_\gamma(a)} |T_m^p(f_1, \dots, f_m)(x)|^q \cdot |x|_p^\beta dx \right)^{\frac{1}{q}} \\
&\quad \le \int_{\mathbb{Q}_p^m} K(1, |y_1|_p, \dots, |y_m|_p) \left(\frac{1}{|B_\gamma(a)|_{\mathrm{H}}^{\alpha(1+\lambda q)}} \int_{B_\gamma(a)} \left| \prod_{i=1}^m f_i(xy_i) \right|^q |x|_p^\beta dx \right)^{\frac{1}{q}} d\mathbf{y}.
\end{aligned}$$

Hölder's inequality yields

$$\begin{aligned}
&\left(\frac{1}{|B_\gamma(a)|_{\mathrm{H}}^{\alpha(1+\lambda q)}} \int_{B_\gamma(a)} |T_m^p(f_1, \dots, f_m)(x)|^q \cdot |x|_p^\beta dx \right)^{\frac{1}{q}} \\
&\quad \le \int_{\mathbb{Q}_p^m} K(1, |y_1|_p, \dots, |y_m|_p) \prod_{i=1}^m \|f_i(\cdot y_i)\|_{B_\gamma(a);q_i,\lambda_i,\beta_i} d\mathbf{y}.
\end{aligned}$$

By the p-adic change of variables and the definition of weighted Morrey norms, we have

$$\begin{aligned}
&\left(\frac{1}{|B_\gamma(a)|_{\mathrm{H}}^{\alpha(1+\lambda q)}} \int_{B_\gamma(a)} |T_m^p(f_1, \dots, f_m)(x)|^q \cdot |x|_p^\beta dx \right)^{\frac{1}{q}} \\
&\quad = \int_{\mathbb{Q}_p^m} K(1, |y_1|_p, \dots, |y_m|_p) \prod_{i=1}^m |y_i|_p^{\alpha\left(\frac{1}{q_i}+\lambda_i\right)-\frac{1}{q_i}-\frac{\beta_i}{q}} \|f_i\|_{y_i B_\gamma(a);q_i,\lambda_i,\beta_i} d\mathbf{y} \\
&\quad \le \int_{\mathbb{Q}_p^m} K(1, |y_1|_p, \dots, |y_m|_p) \prod_{i=1}^m |y_i|_p^{\alpha\left(\frac{1}{q_i}+\lambda_i\right)-\frac{1}{q_i}-\frac{\beta_i}{q}} d\mathbf{y} \cdot \prod_{i=1}^m \|f_i\|_{L^{q_i,\lambda_i}(\mathbb{Q}_p, |x|_p^{\beta_i q_i/q})} \\
&\quad = L_m \prod_{i=1}^m \|f_i\|_{L^{q_i,\lambda_i}(\mathbb{Q}_p, |x|_p^{\beta_i q_i/q})}.
\end{aligned}$$

Thus T_m^p is bounded from $L^{q_1,\lambda_1}(\mathbb{Q}_p, |x|_p^{\beta_1 q_1/q}) \times \cdots \times L^{q_m,\lambda_m}(\mathbb{Q}_p, |x|_p^{\beta_m q_m/q})$ to $L^{q,\lambda}(\mathbb{Q}_p, |x|_p^{\beta})$ and

$$\|T_m^p\|_{L^{q_1,\lambda_1}(\mathbb{Q}_p, |x|_p^{\beta_1 q_1/q}) \times \cdots \times L^{q_m,\lambda_m}(\mathbb{Q}_p, |x|_p^{\beta_m q_m/q}) \to L^{q,\lambda}(\mathbb{Q}_p, |x|_p^{\beta})} \le L_m. \tag{7}$$

For the converse when $\lambda_1 q_1 = \cdots = \lambda_m q_m$ and $\alpha > 0$, we employ functions $\hat{f_i}$ as in Lemma 3.1. We calculate

$$\left(\frac{1}{|B_\gamma(a)|_{\mathrm{H}}^{\alpha(1+\lambda q)}}\int_{B_\gamma(a)}|T_m^p(\hat{f_1},\ldots,\hat{f_m})(x)|^q\cdot|x|_p^\beta dx\right)^{\frac{1}{q}}$$
$$=\left(\frac{1}{|B_\gamma(a)|_{\mathrm{H}}^{\alpha(1+\lambda q)}}\int_{B_\gamma(a)}\left|\int_{\mathbb{Q}_p^m}K(1,|y_1|_p,\ldots,|y_m|_p)\prod_{i=1}^m f_i(xy_i)d\mathbf{y}\right|^q\cdot|x|_p^\beta dx\right)^{\frac{1}{q}}$$

by using the assumption $\lambda_1 q_1 = \cdots = \lambda_m q_m = \lambda q$ and $\alpha > 0$:

$$\begin{aligned}&\left(\frac{1}{|B_\gamma(a)|_{\mathrm{H}}^{\alpha(1+\lambda q)}}\int_{B_\gamma(a)}|T_m^p(\hat{f_1},\ldots,\hat{f_m})(x)|^q\cdot|x|_p^\beta dx\right)^{\frac{1}{q}}\\
&=\int_{\mathbb{Q}_p^m}K(1,|y_1|_p,\ldots,|y_m|_p)\cdot\prod_{i=1}^m|y_i|_p^{\alpha\left(\frac{1}{q_i}+\lambda_i\right)-\frac{1}{q_i}-\frac{\beta_i}{q}}d\mathbf{y}\left(\frac{1}{|B_\gamma(a)|_{\mathrm{H}}^{\alpha(1+\lambda q)}}\int_{B_\gamma(a)}|x|_p^{\alpha+\alpha\lambda q-1}dx\right)^{\frac{1}{q}}\\
&=L_m\cdot\left(\frac{1-p^{-1}}{1-p^{-\alpha(1+\lambda q)}}\right)^{\frac{1}{q}}\\
&=L_m\cdot\prod_{i=1}^m\|\hat{f_i}\|_{L^{q_i,\lambda_i}(\mathbb{Q}_p,|x|_p^{\beta_i q_i/q})}\cdot\frac{1}{(1-p^{-\alpha(1+\lambda q)})^{1/q}}\prod_{j=1}^m\left(1-p^{-\alpha(1+\lambda_j q_j)}\right)^{\frac{1}{q_j}}\\
&=L_m\cdot\prod_{i=1}^m\|\hat{f_i}\|_{L^{q_i,\lambda_i}(\mathbb{Q}_p,|x|_p^{\beta_i q_i/q})}.\end{aligned}$$

Therefore,

$$L_m\le\|T_m^p\|_{L^{q_1,\lambda_1}(\mathbb{Q}_p,|x|_p^{\beta_1q_1/q})\times\cdots\times L^{q_m,\lambda_m}(\mathbb{Q}_p,|x|_p^{\beta_mq_m/q})\to L^{q,\lambda}(\mathbb{Q}_p,|x|_p^\beta)}<\infty. \tag{8}$$

Combining (7) and (8), we complete the proof of Theorem 1.4.

4. Examples

In this section, we discuss our main results with regard to some particular choices of kernels. Actually many classical integral operators are special cases of the integral operator T_m^p if one chooses a suitable function K. We write

$$|(y_1,y_2,\ldots,y_m)|_p=\max(|y_1|_p,|y_2|_p,\ldots,|y_m|_p).$$

For example,

(i) m-linear p-adic Hardy operator

$$R_\otimes^p(f_1,\ldots,f_m)(x)=\frac{1}{|x|_p^m}\int\limits_{|(y_1,y_2,\ldots,y_m)|_p\le|x|_p}f_1(y_1)\cdots f_m(y_m)d\mathbf{y},\quad x\in\mathbb{Q}_p^*,$$

if $K(|x|_p,|y_1|_p,\ldots,|y_m|_p)=|x|_p^{-m}\cdot\chi_{\{|(y_1,\ldots,y_m)|_p<|x|_p\}}(|y_1|_p,\ldots,|y_m|_p)$;

(ii) m-linear nontensorial extension of p-adic Hardy-Littlewood-Pólya operator

$$Q_{\neg\otimes}^p(f_1,\ldots,f_m)(x)=\int_{\mathbb{Q}_p^m}\frac{f_1(y_1)\cdots f_m(y_m)}{\left(\max\{|x|_p,|y_1|_p,\ldots,|y_m|_p\}\right)^m}d\mathbf{y},\quad x\in\mathbb{Q}_p^*,$$

if $K(|x|_p, |y_1|_p, \ldots, |y_m|_p) = \frac{1}{[\max(|x|_p,|y_1|_p,\ldots,|y_m|_p)]^m}$;

(iii) m-linear tensorial extension of p-adic Hardy-Littlewood-Pólya operator

$$Q^p_{\otimes}(f_1, \ldots, f_m)(x) = \int_{\mathbb{Q}_p^m} \frac{f_1(y_1)\cdots f_m(y_m)d\mathbf{y}}{(\max\{|x|_p, |y_1|_p\})\cdots(\max\{|x|_p, |y_m|_p\})}, \quad x \in \mathbb{Q}_p^*,$$

if $K(|x|_p, |y_1|_p, \ldots, |y_m|_p) = \frac{1}{\prod_{i=1}^m \max(|x|_p,|y_i|_p)}$;

(iv) m-linear tensorial p-adic Hilbert operator

$$P^p_{\otimes}(f_1, \ldots, f_m)(x) = \int_{\mathbb{Q}_p^m} \frac{f_1(y_1)\cdots f_m(y_m)d\mathbf{y}}{\prod_{i=1}^m(|x|_p + |y_i|_p)}, \quad x \in \mathbb{Q}_p^*,$$

if $K(|x|_p, |y_1|_p, \ldots, |y_m|_p) = \frac{1}{\prod_{i=1}^m(|x|_p+|y_i|_p)}$;

(v) m-linear nontensorial p-adic Hilbert operator

$$P^p_{\neg\otimes}(f_1, \ldots, f_m)(x) = \int_{\mathbb{Q}_p^m} \frac{f_1(y_1)\cdots f_m(y_m)d\mathbf{y}}{(|x|_p + |y_1|_p + \cdots + |y_m|_p)^m}, \quad x \in \mathbb{Q}_p^*,$$

if $K_1(|x|_p, |y_1|_p, \ldots, |y_m|_p) = \frac{1}{(|x|_p+|y_1|_p+\cdots+|y_m|_p)^m}$.

The proofs of part (ii) of next corollaries are quite similar to the ones of part (i). So, we will only give the proof of part (i) of next corollaries.

Corollary 4.1. (i) *Maintain the assumptions of the Theorem 1.2. Then* $R^p_{\otimes}$ *is bounded from* $L^{q_1}(\mathbb{Q}_p, |x|_p^{\alpha_1 q_1/q}) \times \cdots \times L^{q_m}(\mathbb{Q}_p, |x|_p^{\alpha_m q_m/q})$ *to* $L^q(\mathbb{Q}_p, |x|_p^{\alpha})$ *if and only if* $\alpha_i < q(1 - (1/q_i))$, $i = 1, 2, \ldots, m$. *In addition,*

$$\|R^p_{\otimes}\|_{L^{q_1}(\mathbb{Q}_p,|x|_p^{\alpha_1 q_1/q})\times\cdots\times L^{q_m}(\mathbb{Q}_p,|x|_p^{\alpha_m q_m/q})\to L^q(\mathbb{Q}_p,|x|_p^{\alpha})} = \frac{(1-p^{-1})^m}{\prod_{i=1}^m(1-p^{(1/q_i)+(\alpha_i/q)-1})}.$$

(ii) *Maintain the assumptions of the Theorem 1.4. If* $-\alpha\left(\frac{1}{q_i} + \lambda_i\right) + \frac{1}{q_i} + \frac{\beta_i}{q} < 1$, *then* $R^p_{\otimes}$ *is bounded from* $L^{q_1,\lambda_1}(\mathbb{Q}_p, |x|_p^{\beta_1 q_1/q}) \times \cdots \times L^{q_m,\lambda_m}(\mathbb{Q}_p, |x|_p^{\beta_m q_m/q})$ *to* $L^{q,\lambda}(\mathbb{Q}_p, |x|_p^{\beta})$. *Moreover, if* $\lambda_1 q_1 = \cdots = \lambda_m q_m$ *and* $\alpha > 0$, *then*

$$\|R^p_{\otimes}\|_{L^{q_1,\lambda_1}(\mathbb{Q}_p,|x|_p^{\beta_1 q_1/q})\times\cdots\times L^{q_m,\lambda_m}(\mathbb{Q}_p,|x|_p^{\beta_m q_m/q})\to L^{q,\lambda}(\mathbb{Q}_p,|x|_p^{\beta})} = \frac{(1-p^{-1})^m}{\prod_{i=1}^m(1-p^{\delta_i})},$$

where $\delta_i = -\alpha\left(\frac{1}{q_i} + \lambda_i\right) + \frac{1}{q_i} + \frac{\beta_i}{q} - 1, \ i = 1, \ldots, m.$

Proof. Due to Theorem 1.2, $R^p_{\otimes}$ is bounded on $L^q(\mathbb{Q}_p, |x|_p^{\alpha})$ if and only if

$$\begin{aligned} C_{R_\otimes} &= \int_{\mathbb{Q}_p^m} \chi_{\{|(y_1,\ldots,y_m)|_p<1\}}(|y_1|_p, \ldots, |y_m|_p) \cdot \prod_{k=1}^m |y_k|_p^{-(1/q_k)-(\alpha_k/q)} d\mathbf{y} \\ &= \int_{|(y_1,\ldots,y_m)|_p\le 1} \prod_{k=1}^m |y_k|_p^{-1/q_k-(\alpha_k/q)} d\mathbf{y} < \infty. \end{aligned}$$

The convergence and exact value of this integral has been studied by Wu et al. in [7] and we have that the integral converges when $\alpha_i < q(1 - (1/q_i))$. Furthermore,

$$C_{R_\otimes} = \frac{(1-p^{-1})^m}{\prod_{i=1}^m(1-p^{(1/q_i)+(\alpha_i/q)-1})}.$$

□

We remark that Corollary 4.1 can be located as a passage to the p-adic setting of the results in [4]. It is also interesting to compare Corllary 4.1 with the results [9], where Wu, Mi and Fu worked in central Morrey spaces. See [7, 8] for more about the p-adic Hardy operator.

Corollary 4.2. (i) *Maintain the assumptions of the Theorem 1.2, $Q^p_{\neg\otimes}$ is bounded from $L^{q_1}(\mathbb{Q}_p, |x|_p^{\alpha_1 q_1/q}) \times \cdots \times L^{q_m}(\mathbb{Q}_p, |x|_p^{\alpha_m q_m/q})$ to $L^q(\mathbb{Q}_p, |x|_p^{\alpha})$ if and only if $\alpha_i < q(1-(1/q_i))$ and $-1 < \alpha$. In addition,*

$$\|Q^p_{\neg\otimes}\|_{L^{q_1}(\mathbb{Q}_p,|x|_p^{\alpha_1 q_1/q})\times\cdots\times L^{q_m}(\mathbb{Q}_p,|x|_p^{\alpha_m q_m/q})\to L^q(\mathbb{Q}_p,|x|_p^{\alpha})} = \frac{(1-p^{-1})^m(1-q^{-m})}{(1-p^{-(1/q)-(\alpha/q)})\prod_{i=1}^m(1-p^{(1/q_i)+(\alpha_i/q)-1})}.$$

(ii) *Maintain the assumptions of the Theorem 1.4, if $-\alpha\left(\frac{1}{q_i}+\lambda_i\right)+\frac{1}{q_i}+\frac{\beta_i}{q}<1$ and $\alpha\left(\frac{1}{q}+\lambda\right)-\frac{1}{q}-\frac{\beta}{q}<0$, then $Q^p_{\neg\otimes}$ is bounded from $L^{q_1,\lambda_1}(\mathbb{Q}_p, |x|_p^{\beta_1 q_1/q}) \times \cdots \times L^{q_m,\lambda_m}(\mathbb{Q}_p, |x|_p^{\beta_m q_m/q})$ to $L^{q,\lambda}(\mathbb{Q}_p, |x|_p^{\beta})$. Moreover, if $\lambda_1 q_1 = \cdots = \lambda_m q_m$ and $\alpha > 0$, then*

$$\|Q^p_{\neg\otimes}\|_{L^{q_1,\lambda_1}(\mathbb{Q}_p,|x|_p^{\beta_1 q_1/q})\times\cdots\times L^{q_m,\lambda_m}(\mathbb{Q}_p,|x|_p^{\beta_m q_m/q})\to L^{q,\lambda}(\mathbb{Q}_p,|x|_p^{\beta})} = \frac{(1-p^{-1})^m(1-q^{-m})}{(1-p^{-\delta})\prod_{i=1}^m(1-p^{\delta_i})},$$

where $\delta = -\alpha\left(\frac{1}{q}+\lambda\right)+\frac{1}{q}+\frac{\beta}{q}$ and $\delta_i = -\alpha\left(\frac{1}{q_i}+\lambda_i\right)+\frac{1}{q_i}+\frac{\beta_i}{q}-1,\ \ i=1,\ldots,m.$

Proof. By Theorem 1.2, $Q^p_{\neg\otimes}$ is bounded on $L^q(\mathbb{Q}_p, |x|_p^{\alpha})$ if and only if

$$C_{Q_{\neg\otimes}} = \int_{\mathbb{Q}_p^m} \frac{\prod_{i=1}^m |y_i|^{-(1/q_i)-(\alpha_i/q)}}{|(1,y_1,y_2,\ldots,y_m)|_p^m} d\mathbf{y} < \infty.$$

The convergence and exact value of this integral has been studied by Wu et al. in [7] and we have that the integral converges when $\alpha_i < q(1-(1/q_i))$, $i = 1,2,\ldots,m$ and $-1 < \alpha$. Furthermore,

$$C_{Q_{\neg\otimes}} = \frac{(1-p^{-1})^m(1-q^{-m})}{(1-p^{-(1/q)-(\alpha/q)})\prod_{i=1}^m(1-p^{(1/q_i)+(\alpha_i/q)-1})}.$$

□

See [7, 8] for more about the p-adic Hardy–Littlewood–Polya operator.

Corollary 4.3. (i) *Maintain the assumptions of the Theorem 1.2, $Q^p_{\otimes}$ is bounded from $L^{q_1}(\mathbb{Q}_p, |x|_p^{\alpha_1 q_1/q}) \times \cdots \times L^{q_m}(\mathbb{Q}_p, |x|_p^{\alpha_m q_m/q})$ to $L^q(\mathbb{Q}_p, |x|_p^{\alpha})$ if and only if $0 < (1/q_i) + (\alpha_i/q) < 1$. In addition,*

$$\|Q^p_{\otimes}\|_{L^{q_1}(\mathbb{Q}_p,|x|_p^{\alpha_1 q_1/q})\times\cdots\times L^{q_m}(\mathbb{Q}_p,|x|_p^{\alpha_m q_m/q})\to L^q(\mathbb{Q}_p,|x|_p^{\alpha})} = \left(1-\frac{1}{p}\right)^{2m}\cdot\prod_{i=1}^m \frac{1}{(1-p^{-(1/q_i)-(\alpha_i/q)})(1-p^{(1/q_i)+(\alpha_i/q)-1})}.$$

(ii) *Maintain the assumptions of the Theorem 1.4, if $0 < -\alpha\left(\frac{1}{q_i}+\lambda_i\right)+\frac{1}{q_i}+\frac{\beta_i}{q} < 1$, then $Q^p_{\otimes}$ is bounded from $L^{q_1,\lambda_1}(\mathbb{Q}_p, |x|_p^{\beta_1 q_1/q}) \times \cdots \times L^{q_m,\lambda_m}(\mathbb{Q}_p, |x|_p^{\beta_m q_m/q})$ to $L^{q,\lambda}(\mathbb{Q}_p, |x|_p^{\beta})$. Moreover, if $\lambda_1 q_1 = \cdots = \lambda_m q_m$ and $\alpha > 0$, then*

$$\|Q^p_{\otimes}\|_{L^{q_1,\lambda_1}(\mathbb{Q}_p,|x|_p^{\beta_1 q_1/q})\times\cdots\times L^{q_m,\lambda_m}(\mathbb{Q}_p,|x|_p^{\beta_m q_m/q})\to L^{q,\lambda}(\mathbb{Q}_p,|x|_p^{\beta})} = \left(1-\frac{1}{p}\right)^{2m}\cdot\prod_{i=1}^m \frac{1}{(1-p^{-\delta_i-1})(1-p^{\delta_i})},$$

where $\delta_i = -\alpha\left(\frac{1}{q_i}+\lambda_i\right)+\frac{1}{q_i}+\frac{\beta_i}{q}-1,\ \ i=1,\ldots,m.$

Proof. By Theorem 1.2, $Q_{\otimes}^{p}$ is bounded on $L^q(\mathbb{Q}_p, |x|_p^{\alpha})$ if and only if

$$C_{Q_{\otimes}} = \int_{\mathbb{Q}_p^n} \prod_{i=1}^{m} \frac{|y_i|^{-(1/q_i)-(\alpha_i/q)}}{\max(1, |y_i|_p)} d\mathbf{y} < \infty.$$

Let $S_k = \{y : |y|_p = p^k\}$. Arithmetics shows

$$\begin{aligned}
C_{Q_{\otimes}} &= \int_{\mathbb{Q}_p^m} \prod_{i=1}^{m} \frac{|y_i|_p^{-(1/q_i)-(\alpha_i/q)}}{\max(1, |y_i|_p)} d\mathbf{y} = \prod_{i=1}^{m} \left(\int_{\mathbb{Q}_p} \frac{|y_i|_p^{-(1/q_i)-(\alpha_i/q)}}{\max(1, |y_i|_p)} dy_i \right) \\
&= \prod_{i=1}^{m} \left(\int_{|y_i|_p \geq 1} |y_i|_p^{-(1/q_i)-(\alpha_i/q)-1} dy_i + \int_{|y_i|_p < 1} |y_i|_p^{-(1/q_i)-(\alpha_i/q)} dy_i \right) \\
&= \prod_{i=1}^{m} \left(\sum_{k \geq 1} \int_{S_k} |y_i|_p^{-(1/q_i)-(\alpha_i/q)-1} dy_i + \sum_{k \leq 0} \int_{S_k} |y_i|_p^{-(1/q_i)-(\alpha_i/q)} dy_i \right) \\
&= \prod_{i=1}^{m} \left(\left(1 - \frac{1}{p}\right) \sum_{k \geq 1} p^{k(-(1/q_i)-(\alpha_i/q))} + \left(1 - \frac{1}{p}\right) \sum_{k \leq 0} p^{k-k((1/q_i)+(\alpha_i/q))} \right) \\
&= \left(1 - \frac{1}{p}\right)^m \prod_{i=1}^{m} \left(\sum_{k \geq 1} p^{k(-(1/q_i)-(\alpha_i/q))} + \sum_{k \geq 0} p^{k((1/q_i)+(\alpha_i/q)-1)} \right) \\
&= \left(1 - \frac{1}{p}\right)^m \prod_{i=1}^{m} \left(\sum_{k \geq 1} p^{k(-(1/q_i)-(\alpha_i/q))} + \sum_{k \geq 0} p^{k((1/q_i)+(\alpha_i/q)-1)} \right) \\
&= \left(1 - \frac{1}{p}\right)^{2m} \cdot \prod_{i=1}^{m} \frac{1}{(1 - p^{-(1/q_i)-(\alpha_i/q)})(1 - p^{(1/q_i)+(\alpha_i/q)-1})},
\end{aligned}$$

where the series converge due to $0 < (1/q_i) + (\alpha_i/q) < 1$, $i = 1, 2, \ldots, m$. □

Corollary 4.4. (i) *Maintain the assumptions of the Theorem 1.2, $P_{\otimes}^{p}$ is bounded from $L^{q_1}(\mathbb{Q}_p, |x|_p^{\alpha_1 q_1/q}) \times \cdots \times L^{q_m}(\mathbb{Q}_p, |x|_p^{\alpha_m q_m/q})$ to $L^q(\mathbb{Q}_p, |x|_p^{\alpha})$ if and only if $0 < (1/q_i) + (\alpha_i/q) < 1$, $i = 1, 2, \ldots, m$. In addition,*

$$\|P_{\otimes}^{p}\|_{L^{q_1}(\mathbb{Q}_p, |x|_p^{\alpha_1 q_1/q}) \times \cdots \times L^{q_m}(\mathbb{Q}_p, |x|_p^{\alpha_m q_m/q}) \to L^q(\mathbb{Q}_p, |x|_p^{\alpha})} = \left(1 - \frac{1}{p}\right)^m \cdot \prod_{i=1}^{m} \left(\frac{1}{2} + \sum_{k=1}^{\infty} \left(\frac{p^{-((1/q_i)+(\alpha_i/q)-1)k} + p^{((1/q_i)+(\alpha_i/q))k}}{1 + p^k} \right) \right).$$

(ii) *Maintain the assumptions of the Theorem 1.4, if $0 < -\alpha\left(\frac{1}{q_i} + \lambda_i\right) + \frac{1}{q_i} + \frac{\beta_i}{q} < 1$, then $P_{\otimes}^{p}$ is bounded from $L^{q_1, \lambda_1}(\mathbb{Q}_p, |x|_p^{\beta_1 q_1/q}) \times \cdots \times L^{q_m, \lambda_m}(\mathbb{Q}_p, |x|_p^{\beta_m q_m/q})$ to $L^{q, \lambda}(\mathbb{Q}_p, |x|_p^{\beta})$. Moreover, if $\lambda_1 q_1 = \cdots = \lambda_m q_m$ and $\alpha > 0$, then*

$$\|P_{\otimes}^{p}\|_{L^{q_1, \lambda_1}(\mathbb{Q}_p, |x|_p^{\beta_1 q_1/q}) \times \cdots \times L^{q_m, \lambda_m}(\mathbb{Q}_p, |x|_p^{\beta_m q_m/q}) \to L^{q, \lambda}(\mathbb{Q}_p, |x|_p^{\beta})} = \left(1 - \frac{1}{p}\right)^m \cdot \prod_{i=1}^{m} \left(\frac{1}{2} + \sum_{k=1}^{\infty} \left(\frac{p^{-\delta_i k} + p^{(\delta_i + 1)k}}{1 + p^k} \right) \right),$$

where $\delta_i = -\alpha\left(\frac{1}{q_i} + \lambda_i\right) + \frac{1}{q_i} + \frac{\beta_i}{q} - 1$, $i = 1, \ldots, m$.

Proof. By Theorem 1.2, $P_{\otimes}^{p}$ is bounded on $L^q(\mathbb{Q}_p, |x|_p^{\alpha})$ if and only if

$$C_{P_{\otimes}} = \int_{\mathbb{Q}_p^m} \prod_{i=1}^{m} \frac{|y_i|_p^{-(1/q_i)-(\alpha_i/q)}}{1 + |y_i|_p} d\mathbf{y} < \infty.$$

Since series $\sum_{k=1}^{\infty} p^{-((1/q_i)+(\alpha_i/q))k}$ and $\sum_{k=0}^{\infty} p^{((1/q_i)+(\alpha_i/q)-1)k}$ converge when $0 < (1/q_i) + (\alpha_i/q) < 1$, $i = 1, 2, \dots, m$, we have

$$
\begin{aligned}
C_{P_\otimes} &= \int_{\mathbb{Q}_p^m} \prod_{i=1}^{m} \frac{|y_i|_p^{-(1/q_i)-(\alpha_i/q)}}{1+|y_i|_p} d\mathbf{y} \\
&= \prod_{i=1}^{m} \left(\sum_{1\le k<\infty} \int_{S_k} \frac{|y_i|_p^{-(1/q_i)-(\alpha_i/q)}}{1+|y_i|_p} dy_i + \sum_{-\infty<k\le 0} \int_{S_k} \frac{|y_i|^{-(1/q_i)-(\alpha_i/q)}}{1+|y_i|_p} dy_i \right) \\
&= \left(1-\frac{1}{p}\right)^m \cdot \prod_{i=1}^{m} \left(\sum_{1\le k} \frac{p^{-((1/q_i)+(\alpha_i/q)-1)k}}{1+p^k} + \sum_{k\le 0} \frac{p^{-((1/q_i)+(\alpha_i/q)-1)k}}{1+p^k} \right) \\
&= \left(1-\frac{1}{p}\right)^m \cdot \prod_{i=1}^{m} \left(\sum_{k=1}^{\infty} \frac{p^{-((1/q_i)+(\alpha_i/q)-1)k}}{1+p^k} + \sum_{k=0}^{\infty} \frac{p^{((1/q_i)+(\alpha_i/q))k}}{1+p^k} \right) < \infty.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\|P_\otimes^p\|_{L^{q_1}(\mathbb{Q}_p,|x|_p^{\alpha_1 q_1/q})\times\cdots\times L^{q_m}(\mathbb{Q}_p,|x|_p^{\alpha_m q_m/q})\to L^q(\mathbb{Q}_p,|x|_p^{\alpha})} \\
&\quad = \left(1-\frac{1}{p}\right)^m \cdot \prod_{i=1}^{m} \left(\sum_{k=1}^{\infty} \frac{p^{-((1/q_i)+(\alpha_i/q)-1)k}}{1+p^k} + \sum_{k=0}^{\infty} \frac{p^{((1/q_i)+(\alpha_i/q))k}}{1+p^k} \right) \\
&\quad = \left(1-\frac{1}{p}\right)^m \cdot \prod_{i=1}^{m} \left(\frac{1}{2} + \sum_{k=1}^{\infty} \left(\frac{p^{-((1/q_i)+(\alpha_i/q)-1)k} + p^{((1/q_i)+(\alpha_i/q))k}}{1+p^k} \right) \right).
\end{aligned}
$$

□

Corollary 4.5. (i) *Maintain the assumptions of the Theorem 1.2, $P_{\neg\otimes}^p$ is bounded from $L^{q_1}(\mathbb{Q}_p, |x|_p^{\alpha_1 q_1/q}) \times \cdots \times L^{q_m}(\mathbb{Q}_p, |x|_p^{\alpha_m q_m/q})$ to $L^q(\mathbb{Q}_p, |x|_p^{\alpha})$ if and only if $\alpha_i < q(1-(1/q_i))$ and $-1 < \alpha$. In addition,*

$$\|P_{\neg\otimes}^p\|_{L^{q_1}(\mathbb{Q}_p,|x|_p^{\alpha_1 q_1/q})\times\cdots\times L^{q_m}(\mathbb{Q}_p,|x|_p^{\alpha_m q_m/q})\to L^q(\mathbb{Q}_p,|x|_p^{\alpha})} = C_{P_{\neg\otimes}}.$$

(ii) *Maintain the assumptions of the Theorem 1.4, if $-\alpha\left(\frac{1}{q_i}+\lambda_i\right) + \frac{1}{q_i} + \frac{\beta_i}{q} < 1$ and $\alpha\left(\frac{1}{q}+\lambda\right) - \frac{1}{q} - \frac{\beta}{q} < 0$, then $P_{\neg\otimes}^p$ is bounded from $L^{q_1,\lambda_1}(\mathbb{Q}_p, |x|_p^{\beta_1 q_1/q}) \times \cdots \times L^{q_m,\lambda_m}(\mathbb{Q}_p, |x|_p^{\beta_m q_m/q})$ to $L^{q,\lambda}(\mathbb{Q}_p, |x|_p^{\beta})$. Moreover, if $\lambda_1 q_1 = \cdots = \lambda_m q_m$ and $\alpha > 0$, then*

$$\|P_{\neg\otimes}^p\|_{L^{q_1,\lambda_1}(\mathbb{Q}_p,|x|_p^{\beta_1 q_1/q})\times\cdots\times L^{q_m,\lambda_m}(\mathbb{Q}_p,|x|_p^{\beta_m q_m/q})\to L^{q,\lambda}(\mathbb{Q}_p,|x|_p^{\beta})} = \frac{(1-p^{-1})^m(1-q^{-m})}{(1-p^{-\delta})\prod_{i=1}^{m}(1-p^{\delta_i})},$$

where $\delta = -\alpha\left(\frac{1}{q}+\lambda\right) + \frac{1}{q} + \frac{\beta}{q}$ and $\delta_i = -\alpha\left(\frac{1}{q_i}+\lambda_i\right) + \frac{1}{q_i} + \frac{\beta_i}{q} - 1$, $i = 1, \dots, m$.

Proof. By Theorem 1.2, $P_{\neg\otimes}^p$ is bounded on $L^q(\mathbb{Q}_p, |x|_p^{\alpha})$ if and only if

$$C_{P_{\neg\otimes}} = \int_{\mathbb{Q}_p^m} \frac{\prod_{i=1}^{m} |y_i|_p^{-(1/q_i)-(\alpha_i/q)}}{(1+|y_1|_p+\cdots+|y_m|_p)^m} d\mathbf{y} < \infty.$$

Clearly,

$$|(1, y_1, y_2, \dots, y_m)|_p^m \le (1+|y_1|_p+\cdots+|y_m|_p)^m \le m^m |(1, y_1, y_2, \dots, y_m)|_p^m.$$

Thus, $C_{P_{\neg\otimes}}$ is convergent if and only if

$$\int_{\mathbb{Q}_p^m} \frac{\prod_{i=1}^m |y_i|^{-(1/q_i)-(\alpha_i/q)}}{|(1, y_1, y_2, \ldots, y_m)|_p^m} d\mathbf{y} < \infty.$$

Then, by Corollary 4.2, we have that the operator $P^p_{\neg\otimes}$ is bounded on $L^q(\mathbb{Q}_p, |x|_p^\alpha)$ if and only if $\alpha_i < q(1-(1/q_i))$ and $-1 < \alpha$. Hence,

$$\|P^p_{\neg\otimes}\|_{L^{q_1}(\mathbb{Q}_p,|x|_p^{\alpha_1 q_1/q})\times\cdots\times L^{q_m}(\mathbb{Q}_p,|x|_p^{\alpha_m q_m/q})\to L^q(\mathbb{Q}_p,|x|_p^\alpha)} = C_{P_{\neg\otimes}}.$$

□

References

[1] Ts. Batbold, Y. Sawano, Sharp bounds for *m*-linear Hilbert-type operators on the weighted Morrey spaces, Mathematical Inequalities & Applications 20 (2017) 263–283.

[2] Á. Bényi, T. Oh, Best constants for certain multilinear integral operators, Journal of Inequalities and Applications 2006 (2006) 1–12.

[3] Y. H. Cao and J. Zhou, Morrey spaces for nonhomogeneous metric measure spaces, Abstract and Applied Analysis 2013 (2013) 1–8.

[4] Z. Fu, L. Grafakos, S. Lu, F. Zhao, Sharp bounds for *m*-linear Hardy and Hilbert operators, Houston Journal of Mathematics 38 (2012) 225–244.

[5] H. Li, J. Jin, On a *p*-adic Hilbert-type integral operator and its applications, Journal of Applied Analysis and Computation 10 (2020) 1326–1334.

[6] V. S. Vladimirov, I. V. Volovich, E. I. Zelenov, *p*-adic analysis and mathematical physics. Series on Soviet and East European Mathematics, 1. World Scientific, River Edge, NJ, 1994.

[7] Q. Wu, Z. Fu, Sharp estimates of *m*-linear *p*-adic Hardy and Hardy-Littlewood-Pólya operators, Journal of Applied Mathematics 2011 (2011) 1–20.

[8] Q. Wu, Z. Fu, S. Lu, Sharp estimates of *p*-adic Hardy and Hardy-Littlewood-Pólya operators, Acta Mathematica Sinica (English Series) 29 2013 137—150.

[9] Q. Wu, L. Mi, Z. W. Fu, Boundedness of *p*-adic Hardy operators and their commutators on *p*-adic central Morrey and BMO spaces, Journal of Function Spaces and Applications 2013 (2013) 1–10.