Squeezing of spin-1 quantum states via a one-axis twisting Hamiltonian

Tuguldur Kh. Begzjav[1,2,*] and Girish S. Agarwal[1,3]

[1] *Institute for Quantum Science and Engineering, Department of Physics and Astronomy, Texas A&M University, College Station, Texas 77843, USA*
[2] *Department of Physics, National University of Mongolia, Ulaanbaatar 14200, Mongolia*
[3] *Department of Biological and Agricultural Engineering, Texas A&M University, College Station, Texas 77843, USA*

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Squeezing of quantum states is of great interest due to its application in high-precision measurement that can exceed standard quantum noise limit described by Heisenberg uncertainty relations. Here, we study the squeezing of quantum systems with SU(3) symmetry via a one-axis twisting Hamiltonian and examine the intriguing connections between the squeezing of spin-1/2 and spin-1 systems. There are seven subalgebras of su(3) Lie algebra. All these subalgebras are identical with su(2) Lie algebra but not all of them have the same anticommutation relations as su(2). Interestingly, squeezing parameters corresponding to spin-1 subalgebras depend not only on structure constants but also on anticommutation relations of the subalgebras. Our results are reported for the subalgebras with vanishing anticommutators and nematic squeezing, while in other cases our first-principle calculation recovers known results.

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I. INTRODUCTION

Squeezed states are the states with reduced quantum fluctuations below the standard quantum limit of fluctuations characterized by the coherent state. Generally, the squeezed states are achieved by applying a second-order nonlinear Hamiltonian on the coherent state with minimum uncertainty fluctuations. As a consequence, redistribution of equally shared quantum fluctuations within two quadratures develops reduced quantum noise lower than the standard quantum limit. The importance of the squeezed states can be justified by their wide range of applications; for example, high-precision spectroscopic [1–6] and interferometric measurements [7–10] (e.g., gravitational waves), atomic clocks [11–13], and quantum information processing [14].

The first experimental demonstration of squeezed states is carried out in an electromagnetic field in an optical cavity by nondegenerate four-wave mixing [15], and later squeezed quantum states are demonstrated also in cold atomic ensembles that resemble a spin-1/2 system by various methods such as quantum state transfer, direct interaction of pseudospins, multiple passes of light, and so forth [16–21]. Since then, spin squeezing was used extensively in other contexts such as Bose-Einstein condensates [22] and polarization optics [23]. So far all these applications required only the squeezed states with SU(2) symmetry. The spin-1 squeezed states [24–30] are needed in the context of three-component Bose-Einstein condensates [31–36] and quantum description of the focused electromagnetic fields [23,37–40], which are described by the SU(3) Lie group. In the context of atomic vapors, the SU(3) symmetry has been realized in the experiment done by Davis *et al.* [41] which produces ferromagneticlike interactions between spin-1 atoms. It is clear from these recent developments that SU(3) symmetry is starting to become important in optical physics starting from multicomponent Bose-Einstein condensates to the quantum states of focused electromagnetic fields. It is expected that it will become more and more relevant in future investigations.

For spin 1, there are eight Hermitian, traceless and orthonormal operators analogous to Pauli matrices for spin 1/2. Three of them are just spin operators, whereas the last five operators are quadrupole operators. They form su(3) Lie algebra. Then the convenient way to study the squeezing of spin 1 turns out to be the investigation of su(2) subalgebras of su(3) Lie algebra. Great progress in this research has been made by Yukawa *et al.* and Huang *et al.* [26,29]. They classify subalgebras into two types, namely, type 1 and type 2 according to the absolute value of the structure constants of the subalgebras, and predict squeezing parameters for type-1 and type-2 subalgebras based on the formal structure of subalgebras, especially on the commutation relations, i.e., structure constants.

In this paper, we study squeezed collective states of isolated N spin-1 systems produced by the action of a one-axis twisting Hamiltonian on a coherent state. The idea of squeezed states is based on constructing states for which uncertainties in certain physical variables are lower than those for coherent states where the Heisenberg uncertainty relation is crucial. This involves a set of operators which are closed under commutation relations. Thus, among the elements of the su(3) algebra, we look for sets of operators which satisfy this property. Our procedure is similar to what is the standard in the field of squeezed states.

We investigate the su(2) subalgebras of the su(3) Lie algebra and identify two types of subalgebras: type 1 and type 2. This classification is based on anticommutation relations

*tuguldurb@num.edu.mn

that differ between the spin squeezed states of the identified types of subalgebras. Coincidentally, our identified types of subalgebras exactly coincide with the types of subalgebras obtained by Yukawa *et al.* in Ref. [26]. However, we concentrate here on the anticommutation relations of the subalgebras rather than the structure constants, and our primary concern here is to obtain the squeezing parameters by first-principle calculations, namely, rigorous calculation of the squeezing parameters using individual spin operators instead of the collective operators. As a result of the first-principle calculations, in some cases, we prove that misleading results about the squeezing parameters can be obtained when the similarities between the algebra of spin 1/2 and the subalgebra of spin 1 are taken into consideration without rigorous calculations.

This paper is organized as follows: In Sec. II, we examine the su(2) subalgebras of the su(3) Lie algebra. In the subsequent Secs. III and IV, we consider the squeezed states produced by applying a one-axis twisting Hamiltonian on the coherent and nematic states, respectively. Then, in Sec. V, we examine the squeezed states in a specific case, namely, for two spin-1 particles. In the final section, Sec. VI, we summarize our work. At the end of this paper, the Appendix briefly reviews the squeezing of spin 1/2 to compare it with spin 1.

II. SUBALGEBRAS OF su(3) LIE ALGEBRA

The spin-1 system is a three-level quantum system determined by spin quantum number $s = 1$. Spin operator $\hat{s}_z$ along the z direction has three distinct eigenvalues, $+1$, 0, and -1, and the corresponding eigenstates are denoted by $|1\rangle$, $|0\rangle$, and $|-1\rangle$ forming a three-dimensional spin space. The dynamics of the spin-1 system is rotation in the spin space and represented by the unitary operators in the form of $e^{-i\phi g}$ where g is the group algebra element of the su(3), and ϕ is the rotation angle. The group algebra element g can be uniquely determined by linear superposition of eight independent, Hermitian and traceless generators; a more physical choice of these generators is the three spin-1 operators $\hat{s}_x$, $\hat{s}_y$, and $\hat{s}_z$ that obey commutation relation $[\hat{s}_i, \hat{s}_j] = i\epsilon_{ijk}\hat{s}_k$ where ϵ_{ijk} is a Levi-Civita tensor, and five quadrupole operators $\hat{q}_{3z^2-r^2}$, $\hat{q}_{x^2-y^2}$, $\hat{q}_{xy}$, $\hat{q}_{xz}$, and $\hat{q}_{yz}$ given by

$$\begin{aligned} \hat{q}_{x^2-y^2} &= \hat{s}_x^2 - \hat{s}_y^2, \quad \hat{q}_{3z^2-r^2} = 3\hat{s}_z^2 - 2\hat{I}, \\ \hat{q}_{xy} &= \{\hat{s}_x, \hat{s}_y\}, \quad \hat{q}_{xz} = \{\hat{s}_x, \hat{s}_z\}, \\ \hat{q}_{yz} &= \{\hat{s}_y, \hat{s}_z\}, \end{aligned} \tag{1}$$

where the curly brackets denote anticommutator and $\hat{I}$ is the unit operator.

There are seven subalgebras of the su(3) as listed below [26,34]:

$$\begin{aligned} &(1)\ \hat{s}_x, \hat{s}_y, \hat{s}_z; \\ &(2)\ \hat{s}_x, \hat{q}_{xy}, \hat{q}_{xz}; \\ &(3)\ \hat{s}_y, \hat{q}_{yz}, \hat{q}_{xy}; \\ &(4)\ \hat{s}_z, \hat{q}_{xz}, \hat{q}_{yz}; \\ &(5)\ \hat{s}_x, \hat{q}_{yz}, \hat{q}_+; \\ &(6)\ \hat{s}_y, \hat{q}_{xz}, \hat{q}_-; \\ &(7)\ \hat{s}_z, \hat{q}_{x^2-y^2}, \hat{q}_{xy}, \end{aligned} \tag{2}$$

TABLE I. Anticommutation relations of spin-1 operators for the su(2) subalgebras given by Eq. (2).

Subalgebras	Anticommutation relations
Type 1	
1	$\{\hat{s}_x, \hat{s}_y\} = \hat{q}_{xy}$, $\{\hat{s}_x, \hat{s}_z\} = \hat{q}_{xz}$, $\{\hat{s}_y, \hat{s}_z\} = \hat{q}_{yz}$
2	$\{\hat{s}_x, \hat{q}_{xy}\} = \hat{s}_y$, $\{\hat{s}_x, \hat{q}_{xz}\} = \hat{s}_z$, $\{\hat{q}_{xy}, \hat{q}_{xz}\} = -\hat{q}_{yz}$
3	$\{\hat{s}_y, \hat{q}_{xy}\} = \hat{s}_x$, $\{\hat{s}_y, \hat{q}_{yz}\} = \hat{s}_z$, $\{\hat{q}_{xy}, \hat{q}_{yz}\} = -\hat{q}_{xz}$
4	$\{\hat{s}_z, \hat{q}_{xz}\} = \hat{s}_x$, $\{\hat{s}_z, \hat{q}_{yz}\} = \hat{s}_y$, $\{\hat{q}_{xz}, \hat{q}_{yz}\} = -\hat{q}_{xy}$
Type 2	
5	$\{\hat{s}_x, \hat{q}_{yz}\} = 0$, $\{\hat{s}_x, \hat{q}_+\} = 0$, $\{\hat{q}_{yz}, \hat{q}_+\} = 0$
6	$\{\hat{s}_y, \hat{q}_{xz}\} = 0$, $\{\hat{s}_y, \hat{q}_-\} = 0$, $\{\hat{q}_{xz}, \hat{q}_-\} = 0$
7	$\{\hat{s}_z, \hat{q}_{x^2-y^2}\} = 0$, $\{\hat{s}_z, \hat{q}_{xy}\} = 0$, $\{\hat{q}_{x^2-y^2}, \hat{q}_{xy}\} = 0$

where $\hat{q}_\pm = (\hat{q}_{x^2-y^2} \pm \hat{q}_{3z^2-r^2})/2$. These seven subalgebras are all three-dimensional and satisfy three properties of a Lie algebra, namely, (i) operators form a linear vector space, (ii) operators are closed under the commutation relations, and (iii) operators satisfy the Jacobi identity. The explicit form of commutation relations is

$$[\hat{A}_i, \hat{A}_j] = i\epsilon_{ijk}\hat{A}_k \tag{3}$$

for the first four subalgebras and

$$[\hat{A}_i, \hat{A}_j] = 2i\epsilon_{ijk}\hat{A}_k \tag{4}$$

for the last three subalgebras [26], and here the operators $\hat{A}_i$ stand for a triad of operators of the subalgebras. It should be noted that the seven algebras are not independent; however, independence is not necessary to construct the squeezed states. The construction of squeezed states is guided by the Heisenberg uncertainty relations and thus Eqs. (3) and (4) are important. Further, our construction of SU(3) squeezed states does not exhaust all possible squeezed states for such systems.

Anticommutators are given in Table I. Notice that there are two types of subalgebras. Namely, type 1: the first four subalgebras that have nonzero anticommutators; and type-2: the last three subalgebras that have vanishing anticommutators. For type 1, the anticommutators shown in Table I can be compactly written as

$$\begin{aligned} \{\hat{s}_i, \hat{s}_j\} = \hat{q}_{ij}, \quad \{\hat{s}_i, \hat{q}_{ij}\} = \hat{s}_j, \\ \{\hat{q}_{ij}, \hat{q}_{jk}\} = -\hat{q}_{ik}, \end{aligned} \tag{5}$$

where i, j take values x, y, and z.

For the system composed of many spin-1 particles, collective spin and quadrupole operators are defined as the sum of all individual spin and quadrupole operators. They are given as follows:

$$\begin{aligned} \hat{S}_i = \sum_{\alpha=1}^{N} \hat{s}_{i\alpha}, \quad \hat{Q}_{ij} = \sum_{\alpha=1}^{N} \hat{q}_{ij\alpha}, \quad \hat{Q}_\pm = \sum_{\alpha=1}^{N} \hat{q}_{\pm\alpha}, \\ \hat{Q}_{x^2-y^2} = \sum_{\alpha=1}^{N} \hat{q}_{x^2-y^2,\alpha}, \quad \text{and} \quad \hat{Q}_{3z^2-r^2} = \sum_{\alpha=1}^{N} \hat{q}_{3z^2-r^2,\alpha}, \end{aligned} \tag{6}$$

where N stands for the number of spin-1 particles. For notation, the uppercase operators stand for collective spin operators, whereas the lowercase operators stand for single

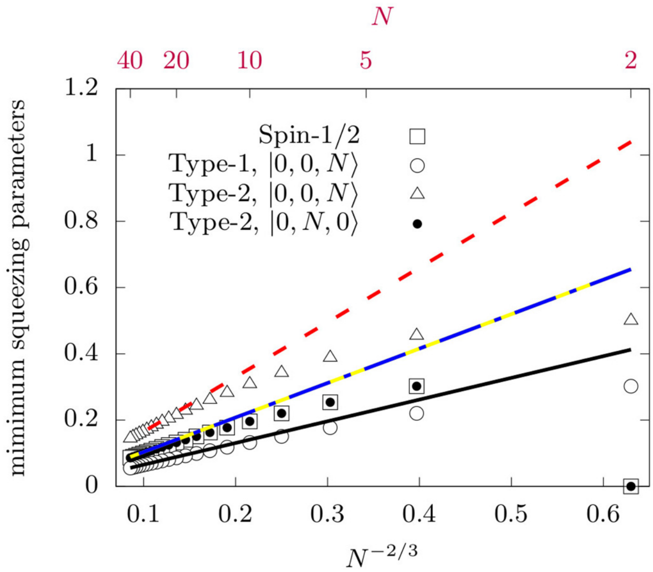

FIG. 1. For large N, there is a linear dependence between the minimum attainable squeezing parameters and $N^{-2/3}$. This dependence is depicted by (i) a solid yellow line for spin 1/2, (ii) a solid black line for type-1 squeezing of coherent state $|0,0,N\rangle$, (iii) a red dashed line for type-2 squeezing of coherent state $|0,0,N\rangle$, and (iv) a blue dash-dotted line for type-2 squeezing of nematic state $|0,N,0\rangle$. Numerically calculated exact values are plotted by rectangles, circles, triangles, and dots.

spin operators; for example, $\hat{s}_{x\alpha}$ denotes the x component of the αth spin operator. This notation is used throughout this paper.

Finally, it is worth mentioning that our classification of su(3) Lie algebra based on anticommutation relations coincidentally coincides with the classification provided in Ref. [26].

In the next two sections, we discuss the spin as well as nematic squeezing of the specifically chosen initial states via a one-axis twisting Hamiltonian where we find very interesting results (see Fig. 1, in particular, the black curve). For the general initial states, the question is very much open even in the context of spin-1/2 systems. Some answers are known for states obtained by rotation of, say, the lowest state. The spin-1 space is much richer and rotations form only a small subgroup.

III. SPIN SQUEEZING OF |0, 0, *N*⟩ STATE VIA A ONE-AXIS TWISTING HAMILTONIAN

In this section, we focus on squeezing parameters of the various types of spin-1 squeezed states produced by applying a one-axis twisting Hamiltonian on the coherent spin state $|0,0,N\rangle$. Note that any state in which all spins are aligned in the same direction would be referred to as a coherent state. Here the notation $|N_1,N_0,N_{-1}\rangle$ means N_1, N_0, and N_{-1} spin-1 particles are in the levels $|1\rangle$, $|0\rangle$, and $|-1\rangle$, respectively. We use the squeezing parameter defined in Ref. [42] throughout this paper. Details of the squeezing parameter are given in the Appendix [also see our Eqs. (10), (20), and (23)].

TABLE II. Expectation values of the collective operators in the state $|0,0,N\rangle$.

Subalgebras	Expectation values
Type 1	
$\hat{S}_x, \hat{S}_y, \hat{S}_z$	$\langle\hat{S}_x\rangle = \langle\hat{S}_y\rangle = 0,\ \langle\hat{S}_z\rangle = -N$
$\hat{S}_x, \hat{Q}_{xy}, \hat{Q}_{xz}$	$\langle\hat{S}_x\rangle = \langle\hat{Q}_{xy}\rangle = \langle\hat{Q}_{xz}\rangle = 0$
$\hat{S}_y, \hat{Q}_{yz}, \hat{Q}_{xy}$	$\langle\hat{S}_y\rangle = \langle\hat{Q}_{yz}\rangle = \langle\hat{Q}_{xy}\rangle = 0$
$\hat{S}_z, \hat{Q}_{xz}, \hat{Q}_{yz}$	$\langle\hat{S}_z\rangle = -N,\ \langle\hat{Q}_{xz}\rangle = \langle\hat{Q}_{yz}\rangle = 0$
Type 2	
$\hat{S}_x, \hat{Q}_{yz}, \hat{Q}_+$	$\langle\hat{S}_x\rangle = \langle\hat{Q}_{yz}\rangle = 0,\ \langle\hat{Q}_+\rangle = N/2$
$\hat{S}_y, \hat{Q}_{xz}, \hat{Q}_-$	$\langle\hat{S}_y\rangle = \langle\hat{Q}_{xz}\rangle = 0,\ \langle\hat{Q}_-\rangle = -N/2$
$\hat{S}_z, \hat{Q}_{x^2-y^2}, \hat{Q}_{xy}$	$\langle\hat{S}_z\rangle = -N,\ \langle\hat{Q}_{x^2-y^2}\rangle = \langle\hat{Q}_{xy}\rangle = 0$

A. Type-2 spin squeezing

First, we consider the squeezed state with respect to the type-2 subalgebras. There are strong similarities between spin-1/2 algebra and type-2 subalgebras; specifically, they not only have similar commutation relations of quadrature operators but also have the same anticommutation relations. Based on these similarities, one can easily conclude the same squeezing parameters for both cases. However, we will reveal here that the squeezing parameter for type-2 spin squeezing is not identical to that of spin 1/2 regardless of their algebraic similarities.

It is sufficient to analyze any of the three subalgebras of type 2 since all squeezing features of the three subalgebras of this type are identical to each other. Therefore, as an example we examine subalgebra 5 (see Table II) in which $\hat{S}_x$ and $\hat{Q}_{yz}$ are quadrature operators orthogonal to the mean spin direction. The well-known Heisenberg uncertainty relation $\langle\Delta\hat{S}_x^2\rangle\langle\Delta\hat{Q}_{yz}^2\rangle \geqslant 4\langle\hat{Q}_+\rangle^2/4 = N^2/4$ leads to the standard quantum limit $N/2$.

The one-axis twisting Hamiltonian is given by $\hat{H}_2 = \chi\hat{S}_x^2$, where χ stands for coupling constant and the label 2 means type-2 subalgebras. Time dynamics is governed by unitary operator $\hat{U}_2(\theta) = \exp(-i\theta\hat{S}_x^2/2)$ with $\theta = 2\chi t$. To obtain the time dynamics of the individual spin operator $\hat{q}_{yz1}$, we calculate the following nested commutation relations:

$$
\begin{aligned}
\left[\hat{S}_x^2, \hat{q}_{yz1}\right] &= 2i\{\hat{s}_{x1}, \hat{q}_{+1}\} + 4i\hat{K}_2\hat{q}_{+1} \\
&= 4i\hat{K}_2\hat{q}_{+1}, \\
\left[\hat{S}_x^2, \left[\hat{S}_x^2, \hat{q}_{yz1}\right]\right] &= 8\hat{K}_2\{\hat{s}_{x1}, \hat{q}_{yz1}\} + 16\hat{K}_2^2\hat{q}_{yz1} \\
&= 16\hat{K}_2^2\hat{q}_{yz1}, \\
\left[\hat{S}_x^2, \left[\hat{S}_x^2, \left[\hat{S}_x^2, \hat{q}_{yz1}\right]\right]\right] &= 32i\hat{K}_2^2\{\hat{s}_{x1}, \hat{q}_{+1}\} + 64i\hat{K}_2^3\hat{q}_{+1} \\
&= 64i\hat{K}_2^3\hat{q}_{+1}, \\
\left[\hat{S}_x^2, \left[\hat{S}_x^2, \left[\hat{S}_x^2, \left[\hat{S}_x^2, \hat{q}_{yz1}\right]\right]\right]\right] &= 128\hat{K}_2^3\{\hat{s}_{x1}, \hat{q}_{yz1}\} + 256\hat{K}_2^4\hat{q}_{yz1} \\
&= 256\hat{K}_2^4\hat{q}_{yz1},
\end{aligned}
\tag{7}
$$

and so on. Here, $\hat{K}_2 = \sum_{\alpha=2}^{N}\hat{s}_{x\alpha}$. The nested commutation relations Eqs. (7) are simple as are Eqs. (A4) since the anticommutators vanish $\{\hat{s}_{x1}, \hat{q}_{+1}\} = \{\hat{s}_{x1}, \hat{q}_{yz1}\} = 0$ as spin 1/2. Furthermore, the time dynamics is obtained as

$$
\hat{U}_2^\dagger\hat{q}_{yz1}\hat{U}_2 = \hat{q}_{yz1}\cos(2\theta\hat{K}_2) - \hat{q}_{+1}\sin(2\theta\hat{K}_2). \tag{8}
$$

This expression looks similar to Eq. (A5) for spin 1/2 given in the Appendix, and the only difference is the double scaling of variable θ. However, the expectation values

$$\langle \hat{Q}_{yz}^2 \rangle = \frac{N}{2} + \frac{N(N-1)}{8}(1-\cos^{2N-4} 2\theta),$$
$$\langle \hat{Q}_{yz}\hat{S}_x \rangle = \langle \hat{S}_x\hat{Q}_{yz} \rangle^* = -i\frac{N}{2}\cos^{2N-2}\theta + \frac{N(N-1)}{4}\sin(2\theta)\cos^{2N-4}\theta \tag{9}$$

look different from those for spin 1/2 given by Eq. (A6). Using Eq. (A2) along with Eqs. (9) and $\langle \hat{S}_x^2 \rangle = N/2$, we obtain the following squeezing parameter (details of the squeezing parameter are given in the Appendix):

$$\xi_2^2(N,\theta) = \frac{\mathrm{var}(\hat{S}_x, \hat{Q}_{yz})}{N/2} = 1 - \frac{1}{8}(N-1) \times [\sqrt{[1-\cos^{2N-4}(2\theta)]^2 + 16\sin^2(2\theta)\cos^{4N-8}(\theta)} - [1-\cos^{2N-4}(2\theta)]]. \tag{10}$$

This squeezing parameter is a result of our first-principle calculations, and it is an exact expression. Furthermore, it is interesting to study the asymptotic behavior of the minimum attainable squeezing parameter and compare it to other cases. For small θ and large N, we approximate the squeezing parameter $\xi_2^2(N,\theta)$ as

$$\xi_2^2(N,\theta) \simeq \frac{1}{16\alpha^2} + \frac{128}{3}\beta^2, \tag{11}$$

where $\alpha = N\theta/4$ and $\beta = N\theta^2/8$ as introduced in Ref. [42]. This approximate squeezing parameter has a minimum value given by

$$\xi_{2,\mathrm{min}}^2(N) \simeq \left(\frac{9}{2}\right)^{1/3} N^{-2/3} \tag{12}$$

at $\theta_0 = (3/4)^{1/6}N^{-2/3}$. To compare the minimum squeezing parameter to other cases, see Fig. 1.

Finally, according to the squeezing parameter Eq. (10), there are no clear relationships between $\xi_{\mathrm{su}(2)}^2$ defined in the Appendix and ξ_2^2, which implies that the similarities of su(2) and type-2 subalgebras of su(3) are deceptive in terms of quantum squeezing. Accordingly, it is inappropriate to analyze the squeezing features just based on the algebraic similarities.

B. Type-1 spin squeezing

In the preceding case, we show that the squeezing properties of the spin-1 subalgebras are not necessary to behave in accordance with their algebraic structures. For this reason, it is worth calculating the squeezing parameter explicitly for type-1 subalgebras, and we present this calculation here.

The case of interest is the type-1 spin squeezing of the $|0,0,N\rangle$ coherent state of N spin-1 systems via a one-axis twisting Hamiltonian. As shown in Table II, the subalgebras 1 and 4 have nonzero expectation values for the state $|0,0,N\rangle$. Therefore, we first consider the subalgebra 1, namely, $(\hat{S}_x, \hat{S}_y, \hat{S}_z)$. The Hamiltonian $\hat{H}_1 = \chi\hat{S}_x^2$, where the label 1 stands for type 1, leads to time dynamics governed by the unitary operator $\hat{U}_1(\theta) = \exp(-i\theta\hat{S}_x^2/2)$, and squeezing quadratures are S_x and S_y. Since $\langle \hat{S}_z \rangle = -N$ (see Table II) the corresponding Heisenberg uncertainty relation is given by $\langle \Delta\hat{S}_x^2 \rangle \langle \Delta\hat{S}_y^2 \rangle \geqslant N^2/4$, and as a result, the standard quantum limit is obtained as $N/2$.

For the subalgebra 1, the physics of the squeezing can be revealed by the clear connection between spin-1/2 and spin-1 particles, i.e., the spin addition rule. In order to see this, we start with two spin-1/2 particles. Then, the squeezed state is described as

$$|\psi\rangle = e^{-i\theta(\hat{s}_{x1}^{(1/2)} \otimes \hat{s}_{x2}^{(1/2)})^2/2}|g\rangle_1 \otimes |g\rangle_2, \tag{13}$$

where $|g\rangle$ is the ground state of the spin-1/2 particles. Here, the operators $\hat{s}_{x1}^{(1/2)}$ and $\hat{s}_{x2}^{(1/2)}$ are x components of individual spin operators of two spin-1/2 particles and they are described in the Hilbert spaces $\mathscr{H}_1^{(1/2)}$ and $\mathscr{H}_2^{(1/2)}$ where $\mathscr{H}_i^{(1/2)}$ is the Hilbert space of the ith -1/2. Now the total Hilbert space $\mathscr{H}_1^{(1/2)} \otimes \mathscr{H}_2^{(1/2)}$ can be decomposed as a direct sum of irreducible subalgebras as follows:

$$\mathscr{H}_1^{(1/2)} \otimes \mathscr{H}_2^{(1/2)} = \mathscr{H}^{(1)} \oplus \mathscr{H}^{(0)}, \tag{14}$$

where $\mathscr{H}^{(1)}$ and $\mathscr{H}^{(0)}$ are Hilbert spaces for spin 1 and spin 0, respectively. This implies that the space of two spin-1/2 particles is identical to the direct sum of spaces for spin-1 and spin-0 particles. As long as the state $|g\rangle_1 \otimes |g\rangle_2$ belongs to $\mathscr{H}^{(1)}$, the squeezed state Eq. (13) can be written as

$$|\psi\rangle = e^{-i\theta\hat{s}_x^2/2}|-1\rangle, \tag{15}$$

where $\hat{s}_x = \hat{s}_{x1}^{(1/2)} + \hat{s}_{x2}^{(1/2)}$ is the x component of a total spin-1 operator that belongs to $\mathscr{H}^{(1)}$, and $|-1\rangle$ denotes $|g\rangle_1 \otimes |g\rangle_2$. Furthermore, Eq. (15) implies that the degree of squeezing of a spin-1 particle is equivalent to that of two spin-1/2 particles when the type-1 subalgebra $(\hat{s}_x, \hat{s}_y, \hat{s}_z)$ is considered. In general, the squeezing parameter of N spin 1 is the same as the squeezing parameter of $2N$ spin-1/2 particles. This argument is first given in Ref. [26] based on the same structure constants of su(2) algebra for spin 1/2 and type-1 subalgebras for spin 1.

Now, we aim to give an explicit calculation of the squeezing parameter of the N spin-1 system and prove that it is the same as the squeezing parameter of the $2N$ spin-1/2 particles. For this purpose, we consider the fourth subalgebra $(\hat{s}_z, \hat{q}_{xz}, \hat{q}_{yz})$ and time evolution operator $\hat{U}_1'(\theta) = \exp(-i\theta\hat{Q}_{xz}^2/2)$ corresponding to the Hamiltonian $\hat{H}_1' = \chi\hat{Q}_{xz}^2$. Here, it is worth mentioning that any subalgebra which belongs to type 1 can represent spin-1 squeezing on an equal footing. The first step of our calculation is to determine the following commutators:

$$\begin{aligned}
\left[\hat{Q}_{xz}^2, \hat{q}_{yz1}\right] &= i\underbrace{\{\hat{q}_{xz1}, \hat{s}_{z1}\}}_{\neq 0} + 2i\hat{K}_1'\hat{s}_{z1} = i\hat{s}_{x1} + 2i\hat{K}_1'\hat{s}_{z1},\\
\left[\hat{Q}_{xz}^2, \left[\hat{Q}_{xz}^2, \hat{q}_{yz1}\right]\right] &= \left(1 + 4\hat{K}_1'^2\right)\hat{q}_{yz1} - 4\hat{K}_1'\hat{q}_{xy1},\\
\left[\hat{Q}_{xz}^2, \left[\hat{Q}_{xz}^2, \left[\hat{Q}_{xz}^2, \hat{q}_{yz1}\right]\right]\right] &= i\left(1 + 12\hat{K}_1'^2\right)\hat{s}_{x1} + i\left(6\hat{K}_1' + 8\hat{K}_1'^3\right)\hat{s}_{z1},\\
\left[\hat{Q}_{xz}^2, \left[\hat{Q}_{xz}^2, \left[\hat{Q}_{xz}^2, \left[\hat{Q}_{xz}^2, \hat{q}_{yz1}\right]\right]\right]\right] &= \left(1 + 24\hat{K}_1'^2 + 16\hat{K}_1'^4\right)\hat{q}_{yz1} - \left(8\hat{K}_1' + 32\hat{K}_1'^3\right)\hat{q}_{xy1},
\end{aligned} \tag{16}$$

and so on. Here, $\hat{K}_1' = \sum_{\alpha=2}^N \hat{q}_{xz\alpha}$. In Eqs. (16), due to nonvanishing anticommutators between $\hat{q}_{xz}$ and each of operators $\hat{s}_{z1}$, $\hat{s}_{x1}$, $\hat{q}_{yz1}$, and $\hat{q}_{xy1}$, the number of terms on the right-hand side increases as the number of nested commutators grows. In general, the nested commutators are given by

$$\underbrace{\left[\hat{Q}_{xz}^2, \left[\hat{Q}_{xz}^2, \left[\ldots\left[\hat{Q}_{xz}^2, \hat{q}_{yz1}\right]\right]\right]\right]}_{n\text{ times}} = \begin{cases} \frac{1}{2}[(1+2\hat{K}_1')^n + (1-2\hat{K}_1')^n]\hat{q}_{yz1} - \frac{1}{2}[(1+2\hat{K}_1')^n - (1-2\hat{K}_1')^n]\hat{q}_{xy1}, & \text{when } n \text{ is even}\\ \frac{i}{2}[(1+2\hat{K}_1')^n + (1-2\hat{K}_1')^n]\hat{s}_{x1} + \frac{i}{2}[(1+2\hat{K}_1')^n - (1-2\hat{K}_1')^n]\hat{s}_{z1}, & \text{when } n \text{ is odd.}\end{cases} \tag{17}$$

This is a complicated pattern as opposed to Eq. (A4) for spin 1/2, and fortunately, we obtain a simple enough equation for the time evolution of $\hat{q}_{yz1}$ as follows:

$$\begin{aligned}
&\hat{U}_1'^{\dagger}\hat{q}_{yz1}\hat{U}_1'\\
&\quad = \hat{q}_{yz1}\cos\left(\frac{\theta}{2}\right)\cos(\theta\hat{K}_1') + \hat{q}_{xy1}\sin\left(\frac{\theta}{2}\right)\sin(\theta\hat{K}_1')\\
&\quad - \hat{s}_{x1}\sin\left(\frac{\theta}{2}\right)\cos(\theta\hat{K}_1') - \hat{s}_{z1}\cos\left(\frac{\theta}{2}\right)\sin(\theta\hat{K}_1').
\end{aligned} \tag{18}$$

Using this time dynamics, the required expectation values can be calculated. The calculation is much more cumbersome but it is straightforward. The results are as follows:

$$\begin{aligned}
\langle\hat{Q}_{yz}^2\rangle &= \frac{N}{2} + \frac{2N(2N-1)}{8}(1 - \cos^{2N-2}\theta),\\
\langle\hat{Q}_{yz}\hat{Q}_{xz}\rangle &= \langle\hat{Q}_{xz}\hat{Q}_{yz}\rangle^* = i\frac{N}{2}\cos^{2N-1}\frac{\theta}{2}\\
&\quad + \frac{2N(2N-1)}{4}\sin\left(\frac{\theta}{2}\right)\cos^{2N-2}\frac{\theta}{2}.
\end{aligned} \tag{19}$$

With the help of Eqs. (19) and $\langle\hat{Q}_{xz}^2\rangle = N/2$, we obtain the squeezing parameter as follows:

$$\begin{aligned}
\xi_1^2(N,\theta) &= \frac{\mathrm{var}(\hat{Q}_{xz}, \hat{Q}_{yz})}{N/2}\\
&= 1 - \frac{1}{4}(2N-1)\\
&\quad \times \left[\sqrt{(1-\cos^{2N-2}\theta)^2 + 16\sin^2\left(\frac{\theta}{2}\right)\cos^{4N-4}\frac{\theta}{2}}\right.\\
&\quad \left. - (1-\cos^{2N-2}\theta)\right].
\end{aligned} \tag{20}$$

This expression implies $\xi_1^2(N,\theta) = \xi_{\mathrm{su}(2)}^2(2N,\theta)$ and it verifies that the statement "squeezing parameters of N spin-1 particles and $2N$ spin-1/2 particles are equal in magnitude."

In the case of large N, an asymptotic form of the minimum attainable squeezing parameter can be found by simply replacing N in Eq. (A8) by $2N$ and it yields

$$\xi_{1,\min}^2(N) \simeq \frac{1}{2}\left(\frac{9}{4}\right)^{1/3} N^{-2/3}. \tag{21}$$

It implies that N spin-1 systems provide approximately 1.58 times stronger squeezing compared with the same number of spin-1/2 particles. The asymptotic form, as well as numerical exact value of the minimum squeezing parameter, are illustrated in Fig. 1.

IV. NEMATIC SQUEEZING OF $|0, N, 0\rangle$ STATE VIA A ONE-AXIS TWISTING HAMILTONIAN

Nematic states of spin 1 with zero spin average $\langle\hat{S}\rangle = 0$ are of great interest and these states do not have spin-1/2 counterparts. For the nematic state $|0, N, 0\rangle$, only two from seven subalgebras have nonzero expectation values (see Table III). These are the subalgebras 5 and 6 of type 2 [see Eq. (2)]. The squeezing of these subalgebras is successfully realized in a Bose-Einstein condensate via nonlinear collisions [34].

Here, we consider the subalgebra 5 as an example, and the nonzero expectation value is $\langle\hat{Q}_+\rangle = -N$. This leads to the uncertainty relation $\langle\Delta\hat{S}_x^2\rangle\langle\Delta\hat{Q}_{yz}^2\rangle \geqslant N^2$ and standard quantum limit N. The corresponding one-axis twisting Hamiltonian is given by $\hat{H}_n = \chi\hat{S}_x^2$, and the time evolution unitary operator is written as $\hat{U}_n(\theta) = e^{-i\theta\hat{S}_x^2/2}$. Here, the label n

TABLE III. Expectation values of the collective operators in the nematic state $|0, N, 0\rangle$.

Subalgebras	Expectation values
Type 1	
$\hat{S}_x, \hat{S}_y, \hat{S}_z$	$\langle\hat{S}_x\rangle = \langle\hat{S}_y\rangle = \langle\hat{S}_z\rangle = 0$
$\hat{S}_x, \hat{Q}_{xy}, \hat{Q}_{xz}$	$\langle\hat{S}_x\rangle = \langle\hat{Q}_{xy}\rangle = \langle\hat{Q}_{xz}\rangle = 0$
$\hat{S}_y, \hat{Q}_{yz}, \hat{Q}_{xy}$	$\langle\hat{S}_y\rangle = \langle\hat{Q}_{yz}\rangle = \langle\hat{Q}_{xy}\rangle = 0$
$\hat{S}_z, \hat{Q}_{xz}, \hat{Q}_{yz}$	$\langle\hat{S}_z\rangle = \langle\hat{Q}_{xz}\rangle = \langle\hat{Q}_{yz}\rangle = 0$
Type 2	
$\hat{S}_x, \hat{Q}_{yz}, \hat{Q}_+$	$\langle\hat{S}_x\rangle = \langle\hat{Q}_{yz}\rangle = 0$, $\langle\hat{Q}_+\rangle = -N$
$\hat{S}_y, \hat{Q}_{xz}, \hat{Q}_-$	$\langle\hat{S}_y\rangle = \langle\hat{Q}_{xz}\rangle = 0$, $\langle\hat{Q}_-\rangle = N$
$\hat{S}_z, \hat{Q}_{x^2-y^2}, \hat{Q}_{xy}$	$\langle\hat{S}_z\rangle = \langle\hat{Q}_{x^2-y^2}\rangle = \langle\hat{Q}_{xy}\rangle = 0$

stands for the nematic squeezing. The time evolution of the operator $\hat{q}_{yz1}$ is already given in the previous section by Eq. (8). The expectation values required for calculation of the squeezing parameter are as follows:

$$\begin{aligned}\langle \hat{S}_x^2 \rangle &= N,\\ \langle \hat{Q}_{yz}^2 \rangle &= N + \frac{N(N-1)}{2}[1-\cos^{N-2}(4\theta)],\\ \langle \hat{Q}_{yz}\hat{S}_x \rangle = \langle \hat{S}_x\hat{Q}_{yz} \rangle^* &= iN\cos^{N-1}(2\theta)\\ &\quad + N(N-1)\sin(2\theta)\cos^{N-2}(2\theta), \end{aligned} \tag{22}$$

and the squeezing parameter is obtained as

$$\begin{aligned}\xi_n^2(N,\theta) &= \frac{\mathrm{var}(\hat{S}_x, \hat{Q}_{yz})}{N}\\ &= 1 - \frac{1}{4}(N-1)\\ &\quad \times [\sqrt{(1-\cos^{N-2}4\theta)^2 + 16\sin^2 2\theta\cos^{2N-4}2\theta}\\ &\quad - (1-\cos^{N-2}4\theta)]. \end{aligned} \tag{23}$$

The fascinating outcome of Eqs. (23) is that the squeezing parameters of the squeezed nematic states of N spin-1 particles are identical to the squeezing parameters of the same number of spin-1/2 particles except for scaling factor 4 along the time axis. The difference in the scaling of time axis t is due to doubled absolute values of the structure constants in Eq. (4). Since the difference is only in the scaling, the minimum values of the squeezing parameters $\xi_n^2(N,\theta)$ are the same as $\xi^2_{\mathrm{su}(2),\min}$ for spin 1/2 given by Eq. (A8). The numerically calculated minimum squeezing parameter is plotted in Fig. 1 by black dots.

V. EXPLICIT FORM OF SQUEEZED STATES FOR $N = 2$

In general, an explicit form of the squeezed states is much more complicated and analytically impossible to obtain. But, it is simple enough for two spin-1 particles. In what follows, we analyze the various types of the squeezed states for $N = 2$.

A. Type-2 squeezing of the state $|0,0,2\rangle$

Using matrix representation we obtain the squeezed state for $N = 2$ as follows:

$$\begin{aligned}&e^{-i\theta\hat{S}_x^2/2}|-1,-1\rangle\\ &\quad = \alpha_1(\theta)|1,1\rangle - \beta(\theta)|1,-1\rangle - 2\beta(\theta)|0,0\rangle\\ &\qquad - \beta(\theta)|-1,1\rangle + \alpha_2(\theta)|-1,-1\rangle, \end{aligned} \tag{24}$$

where we introduce the following functions of θ:

$$\begin{aligned}\alpha_1(\theta) &= \left(e^{-i\theta}\frac{3}{4}\cos\theta - \frac{1}{4}e^{-2i\theta} - \frac{1}{2}e^{-i\theta/2}\right),\\ \alpha_2(\theta) &= \left(e^{-i\theta}\frac{3}{4}\cos\theta - \frac{1}{4}e^{-2i\theta} + \frac{1}{2}e^{-i\theta/2}\right),\\ \beta(\theta) &= \frac{i}{4}e^{-i\theta}\sin\theta. \end{aligned} \tag{25}$$

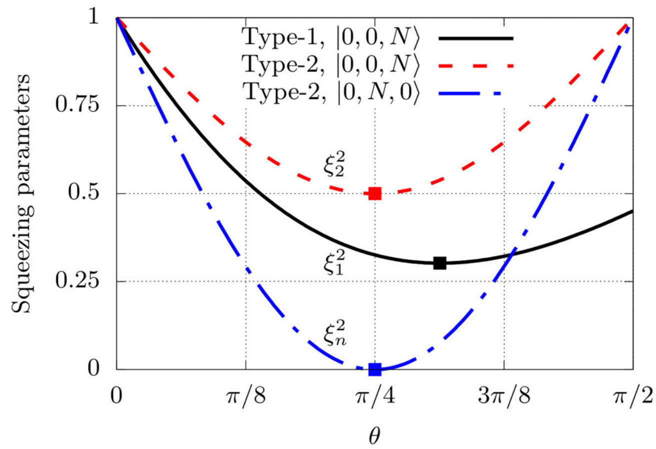

FIG. 2. The squeezing parameters of two spin-1 particles as a function of θ. The minimum attainable values are depicted by rectangles.

Here, the notation $|0,0,2\rangle = |-1\rangle \otimes |-1\rangle = |-1,-1\rangle$ is introduced. For the current case, the squeezing parameter Eq. (10) reads as

$$\xi_2^2(2,\theta) = 1 - \tfrac{1}{2}\sin 2\theta. \tag{26}$$

This is a quite simple function (see Fig. 2), and it has the minimum value $\xi^2_{2,\min}(2) = 1/2$ at the point $\theta_{\min} = \pi/4$. At the minimum point, the squeezed state takes the form

$$\begin{aligned}|\psi_2\rangle = e^{-i\pi\hat{S}_x^2/8}|-1,-1\rangle &= \alpha_1(\pi/4)|1,1\rangle - \beta(\pi/4)|1,-1\rangle\\ &\quad - 2\beta(\pi/4)|0,0\rangle - \beta(\pi/4)|-1,1\rangle\\ &\quad + \alpha_2(\pi/4)|-1,-1\rangle. \end{aligned} \tag{27}$$

This state is obviously entangled, and we quantify the entanglement by von Neumann entropy defined as $S = -\mathrm{Tr}[\rho_1 \ln(\rho_1)]$, where ρ_1 is the density matrix of the first spin-1 particle and Tr( ) stands for trace over the first particle. We obtain the approximate value of S as 0.4135. In comparison to other cases, the squeezed state $|\psi_2\rangle$ is less entangled and squeezed, and this is illustrated by a red rectangle in Fig. 2.

B. Type-1 squeezing of the state $|0,0,2\rangle$

In the case of type-1 squeezing of the coherent state $|0,0,2\rangle$, we obtain the following squeezed state:

$$\begin{aligned}&e^{-i\theta\hat{Q}_{xz}^2/2}|-1,-1\rangle\\ &\quad = \alpha_1(\theta)|1,1\rangle + \beta(\theta)|1,-1\rangle - 2\beta(\theta)|0,0\rangle\\ &\qquad + \beta(\theta)|-1,1\rangle + \alpha_2(\theta)|-1,-1\rangle \end{aligned} \tag{28}$$

with $\alpha_1(\theta)$, $\alpha_2(\theta)$, and $\beta(\theta)$ functions given by Eqs. (25). The corresponding squeezing parameter is derived from Eq. (20) as follows:

$$\xi_1^2(2,\theta) = 1 - \frac{3}{4}(1-\cos^2\theta)\left(\sqrt{\frac{3-\cos\theta}{1-\cos\theta}} - 1\right), \tag{29}$$

and it has minimum value

$$\xi^2_{1,\min}(2) \simeq 0.302\,484 \tag{30}$$

TABLE IV. Relationships between squeezing parameters of spin-1 and spin-1/2 states.

Types of subalgebras	Spin state $|0, 0, N\rangle$	Nematic state $|0, N, 0\rangle$
Type 1	$\xi_1^2(N, \theta) = \xi_{\text{su}(2)}^2(2N, \theta)$ $\xi_{1,\min}^2(N) \simeq (1/2)(9/4)^{1/3} N^{-2/3}$	
Type 2	$\xi_2^2(N, \theta)$, no relationships with $\xi_{\text{su}(2)}^2(N, \theta)$ $\xi_{2,\min}^2(N) \simeq (9/2)^{1/3} N^{-2/3}$	$\xi_n^2(N, \theta) = \xi_{\text{su}(2)}^2(N, 4\theta)$ $\xi_{n,\min}^2(N) \simeq (1/2) 9^{1/3} N^{-2/3}$ This is actually the asymptotic value of minimum squeezing parameter for spin 1/2.

at $\theta_{\min} \simeq 0.983\,04 = 0.312\,911\pi$. The von Neumann entropy of the squeezed state Eq. (29) at $\theta_{\min}$ is obtained as $S \simeq 0.537$.

C. Type-2 squeezing of the state $|0, 2, 0\rangle$

For the nematic state $|0, 2, 0\rangle$, the squeezed state is given by

$$e^{-i\theta \hat{S}_x^2/2}|0, 0\rangle = -2\beta(\theta)(|1, 1\rangle + |1, -1\rangle + 2i\cot\theta|0, 0\rangle + |-1, 1\rangle + |-1, -1\rangle), \quad (31)$$

which is clearly entangled. Equation (23) yields the following simple expression for $N = 2$:

$$\xi_n^2(2, \theta) = 1 - \sin(2\theta), \quad (32)$$

and it has vanishing minimum value $\xi_{n,\min}^2(2) = 0$ at $\theta_{\min} = \pi/4$ (see Fig. 2). The von Neumann entropy of the squeezed state

$$e^{-i\pi \hat{S}_x^2/8}|0, 0\rangle = -\frac{1+i}{4}(|1, 1\rangle + |1, -1\rangle + 2i|0, 0\rangle + |-1, 1\rangle + |-1, -1\rangle) \quad (33)$$

at $\theta_{\min}$ is calculated as $S \simeq 0.6931$, which implies the state Eq. (33) is the most entangled state in comparison to other considered cases. This argument is also justified by the vanishing squeezing parameter depicted by a blue rectangle in Fig. 2.

VI. CONCLUSIONS

We identify seven subalgebras of su(3) algebra that are identical with su(2) algebra in terms of the commutation relations, but four of them (type-1 subalgebras) are different than su(2) spin operators due to their nonvanishing anticommutation relations. The other three subalgebras called type 2 have vanishing anticommutation relations similar to su(2) algebra.

Furthermore, we consider three different instances of squeezing of spin-1 states and do first-principle calculations of squeezing parameters in each instance. We present the results of our calculations (see Table IV) as follows:

(1) Type-1 squeezed states generated by applying a one-axis twisting Hamiltonian on the coherent state $|0, 0, N\rangle$. In this case, we demonstrated the equivalence of squeezing parameters of N spin-1 and $2N$ spin-1/2 particles with the use of first-principle calculations. This result is previously predicted in Ref. [26].

(2) Type-2 squeezed states generated by applying a one-axis twisting Hamiltonian on the coherent state $|0, 0, N\rangle$. Type-2 subalgebra of su(3) and su(2) algebra share not only similar commutation relations but also anticommutation relations. But these similarities are misleading because there are no relationships between the squeezing parameters associated with these two algebras. This result is also proved by rigorous calculations.

(3) Type-2 squeezed states generated by applying a one-axis twisting Hamiltonian on the nematic state $|0, N, 0\rangle$. In this instance, the squeezing parameter is obtained as $\xi_n^2(N, \theta) = \xi_{\text{su}(2)}^2(N, 4\theta)$, which implies that the same number of spin-1 and spin-1/2 particles exhibit the same squeezing properties.

Then, we consider the asymptotic behavior of the squeezing parameters when $N \to \infty$. The results are summarized in Table IV and plotted in Fig. 1. As shown in Fig. 1, the asymptotic results are in agreement with the numerical results, and type-1 spin squeezing of the coherent state $|0, 0, N\rangle$ can achieve stronger squeezing than spin 1/2 when the same number of spin-1 and spin-1/2 particles are considered. However, for type-2 squeezing of the state $|0, 0, N\rangle$ exhibits weaker squeezing compared to spin 1/2. Another situation is the squeezing of the nematic state $|0, N, 0\rangle$. In this case, the minimum attainable squeezing parameter is the same as that of spin 1/2. It is worth mentioning that all results presented in this work are the direct outcome of rigorous first-principle calculations.

Finally, we also examine the specific case $N = 2$ and calculate an explicit form of the squeezed states. The numerical values of the obtained von Neumann entropy verify the close relation between squeezing and entanglement.

We conclude by mentioning future investigations on spin-1 squeezing: (a) to go beyond single twisted squeezed states; (b) to study input states other than the coherent states $|0, 0, N\rangle$ and $|0, N, 0\rangle$; and (c) the decoherence of spin-1 squeezed states under environmental perturbations; the techniques for the latter exist [43].

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APPENDIX: SQUEEZING OF SPIN-1/2 STATES

Squeezing parameters that quantify the spin squeezed states have been studied much more extensively in the case of spin-1/2 particles due to its simplicity. There are several squeezing parameters that are introduced in literature, and the most popular definition has been given by Kitagawa and Ueda in 1993 [42]. They first define mean spin direction as $\mathbf{n} = \langle\hat{S}\rangle/|\langle\hat{S}\rangle|$, where $\hat{S}$ is a collective spin operator of spin 1/2, and then introduce squeezing parameter as a ratio of the minimum variance of quadratures in the plane perpendicular to the mean spin direction to the standard quantum limit characterized by the spin coherent state. Then the squeezing parameter is given as

$$\xi^2_{\mathrm{su}(2)} = \frac{\mathrm{var}(\hat{S}_{\perp 1}, \hat{S}_{\perp 2})}{N/4}, \tag{A1}$$

where $\mathrm{var}(\hat{S}_{\perp 1}, \hat{S}_{\perp 2})$ stands for the minimum variance of two spin quadratures $\hat{S}_{\perp 1}$ and $\hat{S}_{\perp 2}$ in the plane perpendicular to the mean spin direction. In literature, there are two methods to calculate the squeezing parameter Eq. (A1). First, the method used in the seminal paper by Kitagawa and Ueda [42] where two spin quadratures are rotated around the mean spin direction and afterwards one of the variances is minimized by setting a proper rotation angle. This method is based on the collective spin state $|S, M\rangle$ with S being total spin and M being the z component of total spin [42,44]. Second, the method where the minimum and maximum variances are found by means of eigenvalues. This method is based on the number state $|N_+, N_-\rangle$ where N_+ and N_- are populations of excited and ground states of spin 1/2 [45]. The explicit form of the minimum variance found by the second method is as follows:

$$\begin{aligned}&\mathrm{var}(\hat{S}_{\perp 1}, \hat{S}_{\perp 2})\\ &\quad= \tfrac{1}{2}\Big[\big\langle\hat{S}^2_{\perp 1} + \hat{S}^2_{\perp 2}\big\rangle\\ &\qquad - \sqrt{\big\langle\hat{S}^2_{\perp 1} - \hat{S}^2_{\perp 2}\big\rangle^2 + 2\big\langle\hat{S}_{\perp 1}\hat{S}_{\perp 2} + \hat{S}_{\perp 2}\hat{S}_{\perp 1}\big\rangle^2}\Big].\end{aligned} \tag{A2}$$

By calculating the expectation values in Eq. (A2) for a given initial state one can obtain the squeezing parameter $\xi^2_{\mathrm{su}(2)}$ in explicit form.

As an example, we present here calculations of the squeezing parameter of the squeezed state produced by applying a one-axis twisting Hamiltonian $\hat{H} = \chi\hat{S}^2_x$ on coherent spin state $|0, N\rangle$. In this example, spin quadratures are $\hat{S}_{\perp 1} = \hat{S}_x$ and $\hat{S}_{\perp 2} = \hat{S}_y$. We follow the second method given in Ref. [45] since it enables us to understand how the squeezing parameter depends on the corresponding Lie group structure and anticommutation relations. The unitary time-evolution operator at time t becomes $\hat{U}(\theta) = \exp(-i\theta\hat{S}^2_x/2)$ where $\theta = 2\chi t$. Since the operator $\hat{S}_x$ commutes with $\hat{U}(\theta)$, it does not depend on time. Therefore, its expectation value is written as

$$\big\langle\hat{S}^2_x\big\rangle = \frac{N}{4}. \tag{A3}$$

However, $\hat{S}_y$ depends on time. To calculate its time dependence, the following commutation relations are calculated:

$$\begin{aligned}\big[\hat{S}^2_x, \hat{s}_{y1}\big] &= i\{\hat{s}_{x1}, \hat{s}_{z1}\} + 2i\hat{K}\hat{s}_{z1} = 2i\hat{K}\hat{s}_{z1},\\ \big[\hat{S}^2_x, \big[\hat{S}^2_x, \hat{s}_{y1}\big]\big] &= 2\hat{K}\{\hat{s}_{x1}, \hat{s}_{y1}\} + 4\hat{K}^2\hat{s}_{y1}\\ &= 4\hat{K}^2\hat{s}_{y1},\\ \big[\hat{S}^2_x, \big[\hat{S}^2_x, \big[\hat{S}^2_x, \hat{s}_{y1}\big]\big]\big] &= 4i\hat{K}^2\{\hat{s}_{x1}, \hat{s}_{z1}\} + 8i\hat{K}^3\hat{s}_{z1}\\ &= 8i\hat{K}^3\hat{s}_{z1},\\ \big[\hat{S}^2_x, \big[\hat{S}^2_x, \big[\hat{S}^2_x, \big[\hat{S}^2_x, \hat{s}_{y1}\big]\big]\big]\big] &= 8\hat{K}^3\{\hat{s}_{x1}, \hat{s}_{y1}\} + 16\hat{K}^4\hat{s}_{y1}\\ &= 16\hat{K}^4\hat{s}_{y1},\end{aligned} \tag{A4}$$

and so on. Here, $\hat{K} = \sum_{\alpha=2}^{N}\hat{s}_{x\alpha}$. Moreover, in Eq. (A4) we exploit vanishing anticommutation relations between single spin-1/2 operators $\hat{s}_{x1}$, $\hat{s}_{y1}$, and $\hat{s}_{z1}$. These vanishing anticommutators much simplify Eqs. (A4) in contrast to the analogous equation for spin 1 in Sec. III B. With the help of the commutation relations, we obtain the following time evolution for the operator $\hat{s}_{y1}$:

$$\hat{U}^\dagger(\theta)\hat{s}_{y1}\hat{U}(\theta) = \hat{s}_{y1}\cos(\theta\hat{K}) - \hat{s}_{z1}\sin(\theta\hat{K}). \tag{A5}$$

This time dependence of the operator $\hat{s}_{y1}$ gives the expectation values for the operators $\hat{S}^2_y$ and $\hat{S}_y\hat{S}_x$ as follows:

$$\begin{aligned}\big\langle\hat{S}^2_y\big\rangle &= \frac{N}{4} + \frac{N(N-1)}{8}(1 - \cos^{N-2}\theta),\\ \langle\hat{S}_y\hat{S}_x\rangle &= \langle\hat{S}_x\hat{S}_y\rangle^* = i\frac{N}{4}\cos^{N-1}\frac{\theta}{2}\\ &\quad + \frac{N(N-1)}{4}\sin\left(\frac{\theta}{2}\right)\cos^{N-2}\frac{\theta}{2}.\end{aligned} \tag{A6}$$

Plugging Eqs. (A3) and (A6) into Eq. (A2) we obtain the squeezing parameter $\xi^2_{\mathrm{su}(2)}$ as

$$\begin{aligned}&\xi^2_{\mathrm{su}(2)}(N, \theta)\\ &\quad= 1 - \frac{1}{4}(N-1)\\ &\qquad\times\left[\sqrt{(1 - \cos^{N-2}\theta)^2 + 16\sin^2\left(\frac{\theta}{2}\right)\cos^{2N-4}\frac{\theta}{2}}\right.\\ &\qquad\left. - (1 - \cos^{N-2}\theta)\right],\end{aligned} \tag{A7}$$

and it can be found in Ref. [45]. In the case of spin 1/2, the minimum attainable value of the squeezing parameter $\xi^2_{\mathrm{su}(2)}$ for large N is given in Ref. [42] as

$$\xi^2_{\mathrm{su}(2),\min}(N) \simeq \frac{9^{1/3}}{2}N^{-2/3}. \tag{A8}$$

This asymptotic behavior is depicted in Fig. 1 by yellow solid lines, whereas exact values obtained from Eq. (A7) are plotted by square boxes.

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